A Riemannian Approach to Blob Detection in Manifold-Valued Images

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The talk is devoted to generalization of one of the classical image processing methods, blob detection;
In our paper it is generalized for a general setting of an image being a map between manifolds;
The first generalization to such general setting;

The main points of the proposed method:
1. Consider an image graph as a submanifold
2. Define blob response functions by means of image graph curvatures;
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Blob detection

- Blob detection - a widely used keypoints detector for grayscale images ($I: \mathbb{R}^2 \rightarrow \mathbb{R}$);
- Has applications in 3D face recognition, object recognition, panorama stitching, 3D scene modeling, tracking, action recognition, medical images processing, etc.;
- Aims to find "ellipse-like regions" of different sizes with similar intensity inside;
- Blobs are sought as local extremums of a blob response function.
Firstly was proposed for grayscale images $I(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$;

Was generalized for $I(x) : X \rightarrow \mathbb{R}, \dim(X) = 2$;

Several approaches to generalize for $I(x) : \mathbb{R}^2 \rightarrow \mathbb{R}^m$. Are based on conversion to grayscale - don't suit for manifold-valued functions;
Blob detection: the grayscale case

A grayscale image on surface $I(x) : X \to \mathbb{R}$, $\dim(X) = 2$.

1. Calculate the scale-space $L(x, t) : X \times \mathbb{R}_+ \to \mathbb{R}$. $L(x, t)$ is the solution of $\partial_t L(x, t) = -\Delta_{LB} L(x, t)$, $L(x, 0) = I(x)$;

2. Calculate a blob response:

   the determinant blob response: $BR_{\text{det}}(x, t) = \det H_L(x, t)$ or

   the trace blob response: $BR_{\text{tr}}(x, t) = \text{tr} H_L(x, t)$,

   where $H_L$ is the Hessian of $L(x, t)$ as a function of $x$ with fixed $t$;

3. Find blobs centers and scales:

   $C = \{(x, t) = \text{arg extr}_{x,t} \tilde{BR}(x, t)\}$, where $\tilde{BR} = t BR_{\text{tr}}$ (or $t^2 BR_{\text{det}}$);

   Find the blobs radii as $s = \sqrt{2}t$. 
Blob detection: a map between manifolds

General image $I(x) : X \to Y$, $H_L = \nabla dL$, $H_L \in T^*X \otimes T^*X \otimes TY$.

The straightforward generalization:

1. Scale-space can be calculated as the solution of the manifold-valued heat equation. Such PDEs solution methods are out of scope of our work.

2. Blob response calculation. The determinant blob response $BR_{det} = \det H_L$ is not defined.

3. Blobs centers calculation. We can’t find maximums or minimums of the trace blob response because it is not scalar-valued: $BR_{tr} = \text{tr} H_L \in TY$. 
Ideas for the solution

How to define blob response functions for the general case?
The ideas:

1. The image graph $Gr$ is a submanifold immersed in $X \times Y$. The grayscale and the manifold-valued cases differ only by the co-dimension of the embedding.

2. Use the mean and the scalar curvatures as blob responses:
   - Defined for all co-dimensions;
   - Close to the Hessian trace and determinant if the tangent planes to $Gr$ and $X$ are close.
Ideas for the solution

- How to make tangent spaces of $Gr$ and $X$ close?

  The affine transform:
  $Y \rightarrow \mu Y$, such that
  $G_{\mu Y} = \mu G_Y$

- If $r(X)$ (scalar curvature) $\neq 0$ then the blob response will depend on it. How to deal with it?
  Use a "relative" scalar curvature: subtract from the $r(Gr)$ the scalar curvature of the manifold formed by geodesics (obtained by exponential mapping).
The Riemannian blob response

- $Gr_f$ - a graph of $f(x) : X \rightarrow Y$;
- $H_f$ - the Hessian of $f$;
- $\mu f : X \rightarrow \mu Y := X \rightarrow Y \rightarrow \mu Y$;
- A manifold $N$, its submanifold $M$:
  - $h^N_M$ - the mean curvature of $M$;
  - $r^M_M$ the scalar curvature of $M$;
  - $\exp^N_M$ - an exponential map from $T_{mM}$ to $N$;

Definition 1

The scalar blob response:

$$BR_{\text{scalar}} = \lim_{\mu \rightarrow 0} \frac{1}{\mu^2} \left( r_{Gr_{\mu}L} - r_{\exp^{X \times \mu Y}_{Gr_{\mu}L}} \right),$$

the mean blob response:

$$BR_{\text{mean}} = \lim_{\mu \rightarrow 0} \frac{1}{\mu} h^{X \times \mu Y}_{Gr_{\mu}L}.$$
Connection with the Hessian

- $i, j$ (resp. $\alpha, \beta$) - indices for $X$ (resp. for $Y$);
- $\{e_i\}$ (resp. $\{e_\alpha\}$) - an orthonormal basis of $T_X X$ (resp. $T_Y Y$);

**Theorem 1**

Let $H_{ij} = H_L(e_i, e_j)$, $H^\alpha(,) = \langle H_L(,), e_\alpha \rangle_Y$. Then

\[
BR_{\text{scalar}} = \sum_{i,j=1}^{n} \left( \langle H_{ij}, H_{ji} \rangle_Y - \langle H_{ii}, H_{jj} \rangle_Y \right),
\]

\[
BR_{\text{mean}} = \| (\text{tr } H^1, \ldots, \text{tr } H^m) \|_Y.
\]
Corollary 1

Let $\dim(X) = 2$. Then the scalar blob response is equal to the determinant blob response:

$$BR_{\text{scalar}} = BR_{\text{det}},$$

the mean blob response is equal to the trace blob response:

$$BR_{\text{mean}} = BR_{\text{tr}}.$$
Overview of the proof

Maps: \( f(x) : X \rightarrow Y, \tilde{f}(x) : X \rightarrow E = X \times Y, \tilde{f}(x) = (x, f(x)) \).

Bases: \( \{ e'_i = d\tilde{f}(e_i) \} \in T_{\tilde{y}}Gr_f, \{ e'_\alpha : (e'_\alpha, e'_i)_E = 0 \forall i, \forall \alpha \} \in T_{\tilde{y}}(Gr_f)^\perp, \{ e'_i, e'_\alpha \} \in T_{\tilde{y}}E \).

Lemma 1

Let \( u, v \in T_XX \). Let \( \nabla^{\tilde{f}(X)} \) be the connection on \( Gr_f \) induced by the isomorphism \( \tilde{f} \). Let \( II \) be the second fundamental form of the submanifold \( Gr_f \) of \( E \) with respect to the connection \( \nabla^{\tilde{f}(X)} \). Then
\[
H_{\tilde{f}}(u, v) = II(d\tilde{f}(u), d\tilde{f}(v)).
\]

Lemma 2

\[
II_{Gr_f}(e'_i, e'_j) = \sum_{\alpha, \beta = 1}^{m} H_{ij}^{\alpha \beta} g'_{\alpha \beta} e'_\beta.
\]
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Experimental setup

An application: chemical compounds classification problem (the QSAR problem);

- The task: distinguish active and non-active compounds using their structure;
- Compound is represented by a triangulated molecular surface and several physico-chemical and geometrical properties on the surface;
- Input data element can be modeled as a 2-dimensional manifold $X$ with a vector-valued function $f(x) : X \rightarrow \mathbb{R}^m$;
We use Riemannian blob detection for the construction of descriptor vectors. The procedure is the following:

1. Detect blobs by our method in each compound surface;
2. Form pairs of blobs on each surface;
3. Transform the blobs pairs into vectors of fixed length by using the bag of words approach;
The implementation

1. Find $\partial z_j L_i$ by the finite differences approximation, where $z_j$ are the directions from $v$ to its neighbour vertices.

2. Find $dL = (dL_i)$ by solving the overdetermined linear system $dL(Z) = \partial z_j L_i$, $Z$ is a matrix which columns are vectors $z_j$.

3. Find $\nabla^X z_j dL$ for each $j$ as by $\nabla^X z_j dL = P T_x X (\nabla^{\mathbb{R}^3} dL)$. $\nabla^{\mathbb{R}^3} dL$ are found by the finite differences approximation.

4. Find $\nabla^X dL = \{ H_{ij}^\alpha \}$ by solving the overdetermined linear system $\nabla^X dL(Z) = \nabla^X z_j dL$, $Z$ is a matrix which columns are vectors $z_j$.

Calculate $BR_{\text{scalar}}(x, t) = \sum_{\alpha=1}^m \det H^\alpha$, $BR_{\text{mean}}(x, t) = \| \text{tr} \ H^\alpha \|$.
The results

The methods to compare:

1. Riemannian blob detection with $BR_{\text{scalar}}$ as a blob response function;
2. A naive method of applying blob detection to each channel separately;
3. Riemannian blob detection with $BR_{\text{mean}}$ as a blob response function. It coincides with the method, adapted to the case of 2D surface;
4. The method of adaptive neighbourhood projection. It is adapted by us to the case of 2D surface.

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An example of results

Here we can see a molecular surface with $BR_{\text{scalar}}$ on it and found centers (denoted by white color) of blobs of radii 3.
Future work

1. Generalization of our framework to the case of sections of non-trivial fiber bundles. In particular, such generalization will cover an important case of tangent vector fields;

2. Application of the developed method to more practical problems.
Thank you for the attention!