Complexification of Information Geometry in view of quantum estimation theory

by

Hiroshi Nagaoka
Introduction

As H. Shima pointed out in his book:

- $M$: manifold with affine structure (flat connection) $(\theta^i)$

\[ \downarrow \]

$TM$: tangent bundle with a complex structure

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\( TM \) : tangent bundle with a complex structure

and

- \( M \) : manifold with a Hessian structure \(((\theta^i), g, \psi)\)

\[
g_{ij}(\theta) = \partial_i \partial_j \psi(\theta) \quad (\partial = \frac{\partial}{\partial \theta^i})
\]

\[
\downarrow \quad \text{(dually flat structure)}
\]

\( TM \) : tangent bundle with a Kähler structure

with a Kähler potential \( \psi(\theta) \)
A similar situation will appear in the context of quantum estimation theory, where
A similar situation will appear in the context of quantum estimation theory, where

\[ M \] will be replaced with an (classical and quantum) exponential family

and

\[ TM \] will be replaced with the complex projective space (the set of quantum pure states)
Classical Exponential Families

Let

- $\mathcal{X}$: a finite set,
- $\mathcal{P} = \mathcal{P}(\mathcal{X}) := \{p | p : \mathcal{X} \to (0, 1), \sum_{x \in \mathcal{X}} p(x) = 1\}$,
Classical Exponential Families

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- \( \mathcal{P} = \mathcal{P}(\mathcal{X}) := \{ p \mid p : \mathcal{X} \to (0, 1), \sum_{x \in \mathcal{X}} p(x) = 1 \} \),
- \( M = \{ p_\theta \mid \theta \in \Theta(\subset \mathbb{R}^n) \} (\subset \mathcal{P}) \), where

\[
p_\theta(x) = p_0(x) \exp \left[ \sum_{j=1}^{n} \theta^i f_i(x) - \psi(\theta) \right],
\]

\[
\psi(\theta) := \log \sum_{x \in \mathcal{X}} p_0(x) \exp \left[ \sum_{j=1}^{n} \theta^i f_i(x) \right].
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\psi(\theta) := \log \sum_{x \in \mathcal{X}} p_0(x) \exp\left[\sum_{j=1}^{n} \theta^i f_i(x)\right].
\]

We assume

\{1, f_1, \ldots, f_n\} are linearly independent,

which implies

$\theta \mapsto p_\theta$ is injective.
Geometrical Structure of Exponential Family

• Fisher information metric:

\[ g_{ij} = E_\theta[\partial_i \log p_\theta \partial_j \log p_\theta] = \partial_i \partial_j \psi(\theta) \]

( \Rightarrow \text{ Cramér-Rao inequality : } V(\text{estimator}) \geq [g_{ij}]^{-1} )

• e-, m-connections:

<table>
<thead>
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<th>affine coordinates</th>
<th>flat connection</th>
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affine coordinates \quad \text{flat connection}
\begin{align*}
\theta^i & \quad \rightarrow \quad \nabla^{(e)} \\
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\end{align*}

• Duality: \quad (Nagaoka & Amari, 1982)

\[ X g(Y, Z) = g(\nabla_X^{(e)} Y, Z) + g(Y, \nabla_X^{(m)} Z) \]

\Rightarrow (M, g, \nabla^{(e)}, \nabla^{(m)}) \text{ is dually flat}
• $\hat{\eta} := (f_1, \ldots, f_n)$ is an estimator achieving the Cramér-Rao bound (an efficient estimator).
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\mathcal{P} \text{ itself is an exponential family.}
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Quantum State Space

Let $\mathcal{H} \cong \mathbb{C}^d$ be a Hilbert space with an inner product $\langle \cdot | \cdot \rangle$, and define

$$\mathcal{L}(\mathcal{H}) := \{ A | A : \mathcal{H} \xrightarrow{\text{linear}} \mathcal{H} \} = \{\text{linear operators}\},$$

$$\mathcal{L}_h(\mathcal{H}) := \{ A \in \mathcal{L}(\mathcal{H}) | A = A^* \} = \{\text{hermitian operators}\},$$

where $S_r := \{ \rho | \rho \in \mathcal{L}_h(\mathcal{H}) | \rho \geq 0, \text{Tr}[\rho] = 1 \}$.
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$$

- We mainly treat $\mathcal{S}_1$ and $\mathcal{S}_d$ in the sequel.
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Given a manifold \( M = \{ \rho_\theta | \theta = (\theta^i) \in \Theta \} \subset \bar{S} \), let
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- $L_{\theta,i} \in \mathcal{L}_h(\mathcal{H})$ s.t. $\frac{\partial}{\partial \theta^i} \rho_\theta = \frac{1}{2} \left( \rho_\theta L_{\theta,i} + L_{\theta,i} \rho_\theta \right)$

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  Symmetric Logarithmic Derivatives, or SLDs of \( M \)

- \( g_{ij} := \text{Re} \ Tr [\rho_\theta L_{\theta,i} L_{\theta,j}] \).

\[ g = [g_{ij}] \text{ defines a Riemannian metric on } M. \]
SLD Fisher Metric (cont.)

- The metric $g$ is a quantum version of the classical Fisher metric, and is called the SLD metric.
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- The minimum monotone metric. (Petz, 1996)
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- Every $S_r$ becomes a Riemannian space with the SLD metric.
r=d: faithful states

• $S_d = \{ \rho \in \bar{S} \mid \rho > 0 \} = \{ \text{faithful states} \}$. 
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- Since $S_d$ is an open subset in the affine space $\{A \mid A = A^\ast$ and $\text{Tr}A = 1\}$, the $m$-connection $\nabla^{(m)}$ on $S_d$ is defined as the natural flat connection.

$R^{(e)} = 0$ (curvature), $T^{(e)} \neq 0$ (torsion), so $(S_d, g, \nabla^{(e)}, \nabla^{(m)})$ is not dually flat.
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- The e-connection $\nabla^{(e)}$ is defined as the dual of $\nabla^{(m)}$ w.r.t. $g$:

$$X g(Y, Z) = g(\nabla^{(e)}_X Y, Z) + g(Y, \nabla^{(m)}_X Z)$$
\[ r=\mathbb{d} : \text{faithful states} \]

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$r=1$: pure states

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- $S_1 \cong \mathbb{P}(\mathcal{H}) := (\mathcal{H} \setminus \{0\})/\sim$ (complex projective space),
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- The SLD metric $g$ on $\mathcal{S}_1$ coincides with the well-known Fubini-Study metric on $\mathbb{P}(\mathcal{H})$ (up to constant).
\( S_1 \cong \mathbb{P}(\mathcal{H}) \) as a complex manifold

- A \((1, 1)\)-tensor field \( J \) satisfying \( J^2 = -1 \)
  \((\text{almost complex structure})\) is canonically defined by

\[
J \left( \frac{\partial}{\partial x^j} \right) = \frac{\partial}{\partial y^i}, \quad J \left( \frac{\partial}{\partial y^j} \right) = -\frac{\partial}{\partial x^i}
\]

for an arbitrary holomorphic (complex analytic) coordinate system \((z^j) = (x^j + \sqrt{-1}y^j)\).
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- A differential 2-form \( \omega \) is defined by \( \omega(X, Y) = g(X, JY) \).
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- $g$ (or $(J, g, \omega)$) is a Kähler metric in the sense that $\omega$ is a symplectic form: $d\omega = 0$, or equivalently that there is a function called a Kähler potential $f$ satisfying
  $$\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} f.$$
Kahler potential

Let

\[ a_{jk} = g \left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) = g \left( \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k} \right), \]

\[ b_{jk} = g \left( \frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k} \right) = -g \left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k} \right). \]
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Then \( f \) is a Kähler potential iff

\[ a_{jk} = \frac{1}{4} \left( \frac{\partial^2 f}{\partial x^j \partial x^k} + \frac{\partial^2 f}{\partial y^j \partial y^k} \right), \quad \text{and} \]

\[ b_{jk} = \frac{1}{4} \left( \frac{\partial^2 f}{\partial x^j \partial y^k} - \frac{\partial^2 f}{\partial y^j \partial x^k} \right). \]
Quasi-Classical Exponential Family (QCEF)

\[ M = \{ \rho_\theta | \theta \in \mathbb{R}^n \} \subset \bar{S} \] is called a quasi-classical exponential family when it is represented as

\[
\rho_\theta = \exp \left[ \frac{1}{2} \left( \sum_i \theta^i F_i - \psi(\theta) \right) \right] \rho_0 \exp \left[ \frac{1}{2} \left( \sum_i \theta^i F_i - \psi(\theta) \right) \right]
\]

where

\[
\{ F_1, \ldots, F_n \} \subset \mathcal{L}_h(\mathcal{H}),
\]

\[
[F_i, F_j] := F_i F_j - F_j F_i = 0 \quad \text{(commutative),}
\]

\[
\{ \rho_0, F_1 \rho_0, \ldots, F_n \rho_0 \} \quad \text{are linearly independent,}
\]

\[
\psi(\theta) = \log \operatorname{Tr} \left[ \rho_0 \exp \left[ \sum_j \theta^i F_j \right] \right].
\]
Properties of QCEFs

- **e-, m-connections** are defined by

  \[
  \eta_i := \text{Tr}[\rho_\theta F_i] \quad \mapsto \quad \nabla^{(m)}
  \]

  \[
  \theta^i \quad \mapsto \quad \nabla^{(e)}
  \]

- \((M, g, \nabla^{(e)}, \nabla^{(m)})\) is dually flat, where \(g\) is the SLD metric.

- Suppose \(M \subset S^d\). Then \(M\) is e-autoparallel in \(S^d\), and \((g, \nabla^{(e)}, \nabla^{(m)})\) on \(M\) is induced from \((S^d, g, \nabla^{(e)}, \nabla^{(m)})\).

- \((F_1, \ldots, F_n)\) is an estimator for the coordinates \((\eta_1, \ldots, \eta_n)\) achieving the SLD Cramér-Rao bound.

- Since \(\{F_i\}\) are commutative, there exist an orthonormal basis \(\{x^i\}\) with \(X = \{1, 2, \ldots, d\} = \text{dim } H\) and functions \(f_i: X \rightarrow \mathbb{R}\) such that \(F_i = \sum_{x \in X} f_i(x) x^i\).

- Then we have:

  \[
  p_{\theta}(x) := p_0(x) \exp\left[\sum_{i} \theta_i f_i(x) - \psi(\theta)\right] \quad \text{: a classical exponential family}
  \]

  \[
  M = \{\rho_\theta\} \sim \{p_\theta\}\text{ w.r.t. } (g, \nabla^{(e)}, \nabla^{(m)}).\]
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  affine coordinates  \( \theta^i \rightarrow \nabla^{(e)} \)

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Then we have:

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(: a classical exponential family)

and

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M = \{\rho_\theta\} \cong \{p_\theta\} \text{ w.r.t. } (g, \nabla^{(e)}, \nabla^{(m)}).
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Complexification of a pure state QCEF

Let \( M = \{ \rho_\theta \} \) be a quasi-classical exp. family:

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(with the same assumption on \( \{ F_i \} \) as before), and suppose that \( M \subset S_1(\mathcal{H}) \cong \mathbb{P}(\mathcal{H}) \).
Complexification of a pure state QCEF

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For

\( z = (z^1, \ldots, z^n) \in \mathbb{C}^n \), \( z^i = \theta^i + \sqrt{-1} y^i \), \( \theta^i, y^i : \text{real} \),

let

\[
\rho_z := \exp \left[ \frac{1}{2} \left( \sum_i z^i F_i - \psi(\theta) \right) \right] \rho_0 \exp \left[ \frac{1}{2} \left( \sum_i \bar{z}^i F_i - \psi(\theta) \right) \right]
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where

$$U_y := \exp \left[ \frac{\sqrt{-1}}{2} \sum_i y^i F_i \right] : \text{unitary.}$$
Complexification of pure state QCEF (cont.)

Letting \( V \) be a nbd of \( \mathbb{R}^n \) in \( \mathbb{C}^n \) for which \( V \ni z \mapsto \rho_z \) is injective, define

\[
\tilde{M} := \{ \rho_z \mid z \in V \} \quad (\supset M = \{ \rho_\theta \mid \theta \in \mathbb{R}^n \}).
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Complexification of pure state QCEF (cont.)

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Complexification of pure state QCEF (cont.)

Letting $V$ be a nbd of $\mathbb{R}^n$ in $\mathbb{C}^n$ for which $V \ni z \mapsto \rho_z$ is injective, define

$$\tilde{M} := \{\rho_z \mid z \in V\} \quad (\supset M = \{\rho_\theta \mid \theta \in \mathbb{R}^n\}).$$
Complexification of pure state QCEF (cont.)

• \( \tilde{M} \) is a complex (holomorphic) submanifold of \( S_1 \) with a holomorphic coordinate system \((z^i)\), and hence is Kähler w.r.t. \( g_{\tilde{M}} = (\text{Fubini-Study})|_{\tilde{M}} \).
Complexification of pure state QCEF (cont.)

- $\tilde{M}$ is a complex (holomorphic) submanifold of $S_1$ with a holomorphic coordinate system $(z^i)$, and hence is Kähler w.r.t. $g_{\tilde{M}} = (\text{Fubini-Study})|_{\tilde{M}}$.

- When $n = d - 1$, $\tilde{M}$ is open in $S_1$. 

Complexification of pure state QCEF (cont.)

- \( \tilde{M} \) is a complex (holomorphic) submanifold of \( S_1 \) with a holomorphic coordinate system \((z^i)\), and hence is Kähler w.r.t. \( g_{\tilde{M}} = (\text{Fubini-Study})|_{\tilde{M}} \).

- When \( n = d - 1 \), \( \tilde{M} \) is open in \( S_1 \).

- \( 4\psi(\theta) \) gives a Kähler potential on \( \tilde{M} \):

\[
\omega_{\tilde{M}} := \omega|_{\tilde{M}} = 2\sqrt{-1}\partial\bar{\partial}\psi.
\]

Similar to the case of Shima's observation on \( M \) and \( TM \)
Complexification of pure state QCEF (cont.)

- $(\tilde{M}, \eta_i, y^i)$ forms a Darboux coordinate system:

$$\omega_{\tilde{M}} = \sum_{i=1}^{n} d\eta_i \wedge dy^i.$$
Complexification of pure state QCEF (cont.)

- $(\tilde{M}, \eta_i, y^i)$ forms a Darboux coordinate system:

$$\omega_{\tilde{M}} = \sum_{i=1}^{n} d\eta_i \wedge dy^i.$$

- Letting $\nabla^{(m)}$ be the flat connection with affine coordinates $(\eta_i; y^i)$ and $\nabla^{(e)}$ be its dual w.r.t. $g_{\tilde{M}}$. 
• \((\tilde{M}, \eta_i, y^i)\) forms a Darboux coordinate system:

\[
\omega_{\tilde{M}} = \sum_{i=1}^{n} d\eta_i \wedge dy^i.
\]

• Letting \(\nabla^{(m)}\) be the flat connection with affine coordinates \((\eta_i; y^i)\) and \(\nabla^{(e)}\) be its dual w.r.t. \(g_{\tilde{M}}\),

\[
\nabla^{(e)} \circ J = J \circ \nabla^{(m)} \quad \text{and} \quad \nabla^{(e)} \omega_{\tilde{M}} = \nabla^{(m)} \omega_{\tilde{M}} = 0.
\]
Relation to parallel displacement

\[
duality \iff \forall X, Y, Z, \quad X g(Y, Z) = g(\nabla_X^{(e)} Y, Z) + g(Y, \nabla_X^{(m)} Z)
\]
duality \iff \forall X, Y, Z, \ X g(Y, Z) = g(\nabla^{(e)}_X Y, Z) + g(Y, \nabla^{(m)}_X Z)

\[
g(X, Y) = g(X', Y')
\]

if \( X \xrightarrow{e} X' \) and \( Y \xrightarrow{m} Y' \)
Relation to parallel displacement (cont.)

\[ \nabla^{(e)} \circ J = J \circ \nabla^{(m)} \iff \forall X, Y, \ \nabla_{X}^{(e)} J(Y) = J(\nabla_{X}^{(m)} Y) \]
Relation to parallel displacement (cont.)

$$\nabla^{(e)} \circ J = J \circ \nabla^{(m)} \iff \forall X, Y, \quad \nabla^{(e)}_X J(Y) = J(\nabla^{(m)}_X Y)$$

\[
\begin{aligned}
X & \xrightarrow{e} X' \\
\downarrow J & \quad \downarrow J \\
J(X) & \xrightarrow{m} J(X')
\end{aligned}
\]

\[
\begin{aligned}
X & \xrightarrow{m} X' \\
\downarrow J & \quad \downarrow J \\
J(X) & \xrightarrow{e} J(X')
\end{aligned}
\]
Relation to parallel displacement (cont.)

\[ \nabla^{(e)} \omega = \nabla^{(m)} \omega = 0 \iff \forall X, Y, Z \]

\[ X \omega(Y, Z) = \omega(\nabla^{(e)}_X Y, Z) + \omega(Y, \nabla^{(e)}_X Z) \]

\[ = \omega(\nabla^{(m)}_X Y, Z) + \omega(Y, \nabla^{(m)}_X Z) \]
Relation to parallel displacement (cont.)

\[ \nabla^{(e)} \omega = \nabla^{(m)} \omega = 0 \iff \forall X, Y, Z \]

\[ X \omega(Y, Z) = \omega(\nabla^{(e)}_X Y, Z) + \omega(Y, \nabla^{(e)}_X Z) \]
\[ = \omega(\nabla^{(m)}_X Y, Z) + \omega(Y, \nabla^{(m)}_X Z) \]

\[ \omega(X, Y) = \omega(X', Y') \]

if

\[ X \xrightarrow{e} X' \quad \text{and} \quad Y \xrightarrow{e} Y' \]

or

\[ X \xrightarrow{m} X' \quad \text{and} \quad Y \xrightarrow{m} Y' \]
Thank you for listening.