Isometric Reeb Flow and Related Results on Hermitian Symmetric Spaces of Rank 2

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Hereafter let us note that HSSP means Hermitian Symmetric Space.

- **HSSP** of compact type with rank 1: $\mathbb{C}P^m$, $\mathbb{Q}P^m$
- **HSSP** of noncompact type with rank 1: $\mathbb{C}H^m$, $\mathbb{Q}H^m$.
- **HSSP** of compact type with rank 2: $SU(2 + q)/S(U(2) \times U(q))$, $Q^m$, $SO(8)/U(4)$, $Sp(2)/U(2)$ and $(\varepsilon_6(-78), \mathbb{G} \mathbb{O}(10) + \mathbb{R})$
- **HSSP** of compact type with rank 2: $SU(2, q)/S(U(2) \times U(q))$, $Q^*_m$, $SO^*(8)/U(4)$, $Sp(2, \mathbb{R})/U(2)$ and $(\varepsilon_6(2), \mathbb{G} \mathbb{O}(10) + \mathbb{R})$ (See Helgason [6], [7]).
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Let $M$ be a hypersurfaces in a Hermitian Symmetric Space $\tilde{M}$ with Kaehler structure $J$.

$AX = -\tilde{\nabla}_X N$: Weingarten formula

Here $A$: the shape operator of $M$ in $\tilde{M}$.

$\xi = -JN$: the Reeb vector field.

$JX = \phi X + \eta(X)N, \nabla_X \xi = \phi AX$

for any vector field $X \in \Gamma(M)$.

Then $(\phi, \xi, \eta, g)$: almost contact structure on a hypersurface $M$.
A hypersurface $M$: Isometric Reeb Flow $\iff \mathcal{L}_\xi g = 0 \iff g(d\phi_t X, d\phi_t Y) = g(X, Y)$ for any $X, Y \in \Gamma(M)$, where $\phi_t$ denotes a one parameter group, which is said to be an isometric Reeb flow of $M$, defined by

$$\frac{d\phi_t}{dt} = \xi(\phi_t(p)), \quad \phi_0(p) = p, \dot{\phi}_0(p) = \xi(p).$$

Note)

$$\mathcal{L}_\xi g = 0 \iff \nabla_j \xi_i + \nabla_i \xi_j = 0, \nabla \xi: \text{skew-symmetric} \iff g(\nabla X \xi, Y) + g(\nabla Y \xi, X) = 0 \iff g((\phi A - A\phi) X, Y) = 0 \text{ for any } X, Y \in \Gamma(M).$$
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Isometric Reeb Flow on Hermitian Symmetric Spaces
In the future homogeneous hypersurfaces in HSSP satisfying certain geometric conditions might be solved completely as follows:

**Problem 1**
Classify all of homogeneous hypersurfaces in HSSP.

In this talk let us consider hypersurfaces with isometric Reeb flow in *Hermitian Symmetric Spaces* as follows:

**Problem 2**
If $M$ is a complete hypersurface in HSSP $\tilde{M}$ with isometric Reeb flow, then $M$ becomes homogeneous?
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Note 1) In $\mathbb{C}P^m$, $\mathbb{C}H^m$ and $\mathbb{Q}P^m$ with isometric Reeb flow (See Okumura 1976, Montil and Romero 1986, Perez and Martinez 1986).

Note 2) In $G_2(\mathbb{C}^{m+2})$, $G_2^*(\mathbb{C}^{m+2})$ and complex quadric $Q^m = SO(m+2)/SO(2)SO(m)$ with isometric Reeb flow (See Berndt and Suh, 2002 and 2012, Suh, 2013, Berndt and Suh, 2013).

Note 3) In near future, in noncompact complex quadric $Q^{m*} = SO(2, m)/SO(2)SO(m)$ with isometric Reeb flow will be classified.
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In $\mathbb{CP}^m$: Okumura

$M$: an open part of a tube over $\mathbb{CP}^k$, $0 \leq k \leq m - 1$

$H = S(U(k + 1) \cdot U(m - k)) \hookrightarrow SU(m + 1)$

: Isometry group acting cohomogeneity one

$\mathbb{CP}^k = SU(k + 1)/S(U(k) \cdot U(1))$
Montiel and Romero classified hypersurfaces in $\mathbb{C}H^{m}$ with isometric Reeb flow as follows:

**Theorem 1.1**

(Montiel and Romero 1986) Let $M$ be a real hypersurfaces in $\mathbb{C}H^{m}$ with **isometric Reeb flow**. Then we have the following

- (A) $M$ is an open part of a tube around a totally geodesic $\mathbb{C}H^{k}$ in $\mathbb{C}H^{m}$,
- (C) geodesic hypersphere,
- (D) horosphere.
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- (C) geodesic hypersphere,
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Grassmannians

\[ G_2(\mathbb{C}^{m+1}) = SU(m+1)/S(U(m-1) \cdot U(2)) \]

\( M \) : an open part of a tube over \( G_2(\mathbb{C}^{m+1}) \) or over \( \mathbb{QP}^n \), \( m = 2n \)

\( H = S(U(m+1) \cdot U(1)) \hookrightarrow SU(m+2) \)

or \( H = SP(n+1) \hookrightarrow SU(m+2) \)

\[ \mathbb{QP}^n = SP(n+1)/SP(n) \cdot SP(1) \]
When the maximal complex subbundle $\mathcal{C}$ (resp. quaternionic subbundle) of $M$ in $G_2(\mathbb{C}^{m+2})$ is invariant, that is $A\mathcal{C} \subset \mathcal{C}$ (resp. $A\mathcal{Q} \subset \mathcal{Q}$), we say $M$ is Hopf (resp. curvature adapted).

Berndt and Suh (Monat, 1999) have classified real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ as follows:

**Theorem 1.2**

A real hypersurface of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, is Hopf and curvature adapted if and only if it is congruent to

- (A) a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$,
- (B) a tube over a totally geodesic totally real $\mathbb{Q}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$. 

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Isometric Reeb Flow on Hermitian Symmetric Spaces
When the maximal complex subbundle $\mathcal{C}$ (resp. quaternionic subbundle) of $M$ in $G_2(\mathbb{C}^{m+2})$ is invariant, that is $AC \subset \mathcal{C}$ (resp. $AQ \subset \mathcal{Q}$), we say $M$ is Hopf (resp. curvature adapted).

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Berndt and Suh (Monat. 2002) have given a classification of hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with isometric Reeb flow as follows:

**Theorem 1.3**

Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with isometric Reeb flow. Then $M$ is locally congruent to

- (A) a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. The two singular orbits are totally geodesically embedded $\mathbb{C}P^m$ and $G_2(\mathbb{C}^{m+1})$. 
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- Proof of Main Theorem
The Riemannian symmetric space $SU(2, m)/S(U(2) \times U(m))$ is a connected, simply connected, irreducible Riemannian symmetric space of noncompact type with rank 2.

Let $G = SU(2, m)$ and $K = S(U(2) \times U(m))$, and denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebra. Let $B$ denotes the Cartan Killing form of $\mathfrak{g}$ and by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $B$. 
The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g} = \mathfrak{su}(2, m)$. The Cartan involution $\theta \in \text{Aut}(\mathfrak{g})$ on $\mathfrak{su}(2, m)$ is given by $\theta(A) = l_{2,m} A l_{2,m}$ for $A \in \mathfrak{su}(2, m)$, where

$$l_{2,m} = \begin{pmatrix} -l_2 & 0_{2,m} \\ 0_{m,2} & l_m \end{pmatrix}$$

Then $<X, Y> = -B(X, \theta Y)$: a positive definite $\text{Ad}(K)$-invariant on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$: a Riemannian metric $g$, where $g$: the Killing metric on $SU(2, m)/S(U(2) \times U(m))$. 
The Killing Cartan form $B(X,Y)$ of $\mathfrak{sl}(n,C)$ is given by $B(X,Y) = 2nTrXY$ for any $X,Y \in \mathfrak{sl}(n,C)$.

In $\mathfrak{su}(m+2) = \{X \in M(m+2,C)|X^* + X = 0, TrX = 0\}$, $B(X,Y)$ is negative definite, because $B(X,X) = -2nTrXX^* \leq 0$. So $\langle X, Y \rangle = -B(X,Y)$.

In $\mathfrak{su}(2,m) = \{X \in M(m+2,C)|X^*I_{2,m} + I_{2,m}X = 0, TrX = 0\}$, the product $\langle X, Y \rangle = -B(X,\theta Y), \theta^2 = I$, is positive definite, because

$$\langle X, X \rangle = -2nTrX\theta X = -2nTrXI_{2,m}XI_{2,m}$$
$$= 2nTrXX^*I_{2,m}^2 = 2nTrXX^* \geq 0.$$
Killing Cartan forms related to $\mathfrak{sl}(n, C)$

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In $\mathfrak{su}(m + 2) = \{X \in M(m + 2, C) | X^* + X = 0, \text{Tr}X = 0\}$, $B(X, Y)$ is negative definite, because $B(X, X) = -2n \text{Tr}XX^* \leq 0$. So $\langle X, Y \rangle = -B(X, Y)$.

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$$\langle X, X \rangle = -2n \text{Tr}X \theta X = -2n \text{Tr}X I_{2,m} X I_{2,m}$$
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$$
Let \( C = \{ X \in TM | JX \in TM \} \) : the maximal complex subbundle and \( Q = \{ X \in TM | \Im X \subset TM \} \) the maximal quaternionic subbundle for \( M \) in \( SU(2,m)/S(U(2) \times U(m)) \).

When \( C \) and \( Q \) of \( TM \) are both invariant by the shape operator \( A \) of \( M \), we write

\[
h(C, C^\perp) = 0 \quad \text{and} \quad h(Q, Q^\perp) = 0,
\]

where \( h \) denotes the second fundamental form defined by

\[
g(h(X, Y), N) = g(AX, Y)
\]

for any \( X, Y \) on \( M \).
By using the theory of **Focal points** and the method due to P.B. Eberlein, Berndt and Suh proved the following (See Int. J. Math., 2012)

**Theorem 2.1**

Let $M$ be a connected hypersurface in $SU_{2,m}/S(U_{2}U_{m})$, $m \geq 2$. Then $h(C, C^\perp) = 0$ and $h(Q, Q^\perp) = 0$ if and only if $M$ is congruent to an open part of the following:

1. (A) a tube around a totally geodesic $SU_{2,m-1}/S(U_{2}U_{m-1})$ in $SU_{2,m}/S(U_{2}U_{m})$, or

2. (B) a tube around a totally geodesic $HH^n$ in $SU_{2,m}/S(U_{2}U_{m})$, $m = 2n$,

3. (C) a horosphere whose center at infinity is singular.
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(B) a tube around a totally geodesic $HH^n$ in $SU_{2,m}/S(U_2 U_m)$, $m = 2n$,

(C) a horosphere whose center at infinity is singular.
Let $H_t = \cos t e_1 + \sin t e_2 \in \mathfrak{A}$: a unit normal to a horosphere $M_t$, where $\mathfrak{A}$ denotes a maximal abelian subspace of $\mathfrak{P}$ for the E. Cartan’s decomposition $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$.

Here a horosphere is given by $M_t = S_{H_t} \cdot o$, where $S_{H_t}$ denotes the Lie subgroup of $G$ corresponding to the Lie subalgebra $\mathfrak{G}_H = \mathfrak{S} \ominus \mathfrak{R}H$, $\mathfrak{S} = \mathfrak{A} \oplus \mathfrak{N}$ and $\mathfrak{N} = \oplus_{\lambda \in \Sigma} \mathfrak{G}_\lambda$ for the Iwasawa decomposition $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{A} \oplus \mathfrak{N}$ with corresponding $G = KAN$.

The shape operator of a horosphere $M_t$ is given by

$$A_H = \text{ad}(H).$$
Weyl Chamber

\[ \Lambda = \{ \alpha_1, \alpha_2 \} : \text{a set of simple roots} \]

\[ \overline{C}^+(\Lambda) = \{ x \in \mathbb{A} | \langle x, \alpha_i \rangle \geq 0, i = 1,2 \} \]
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Isometric Reeb Flow on Hermitian Symmetric Spaces
Introduction

Hyperbolic Grassmannians

Complex Quadrics

Hypersurfaces in $SU_k, m / R(U_k U_m)$

Isometric Reeb Flow

A point $z$ at infinity defines a foliation of $H^n$ by parallel horospheres.

$H^n = SO^o(n, 1)/SO(n)$
1. Introduction
   - Homogeneous Hypersurfaces
   - Isometric Reeb Flow

2. Hyperbolic Grassmannians
   - Hypersurfaces in $SU_{2,m}/S(U_2 U_m)$
   - Isometric Reeb Flow

3. Complex Quadrics
   - Real hypersurfaces in $Q^{2k}$
   - Tubes around the totally geodesic $\mathbb{CP}^k \subset Q^{2k}$
   - Proof of Main Theorem
In this subsection we introduce a classification with \textit{isometric Reeb flow} in $SU_{2,m}/S(U_2 U_m)$ as follows (See Suh, Advances in Applied Math., 2013):

\textbf{Theorem 2.5}

Let $M$ be a connected orientable real hypersurface in $SU_{2,m}/S(U_2 U_m)$, $m \geq 3$. Then the Reeb flow on $M$ is \textit{isometric} if and only if $M$ is congruent to an open part of the following:

- (A) a tube around some totally geodesic $SU_{2,m−1}/S(U_2 U_{m−1})$ in $SU_{2,m}/S(U_2 U_m)$ or,
- (C) a horosphere whose center at infinity is singular.
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Characterization of type (B) and a Horosphere

Definition
For a real hypersurface $M$ in $SU_{2,m}/S(U_2U_m)$ is said to be a contact $\iff \exists$ a non-zero constant function $\rho$ defined on $M$ such that

$$\phi A + A\phi = k\phi, \quad k = 2\rho,$$

The condition is equivalent to

$$g((\phi A + A\phi)X, Y) = 2d\eta(X, Y),$$

where $d\eta$ is defined by

$$d\eta(X, Y) = (\nabla_X\eta)Y - (\nabla_Y\eta)X$$

for any $X, Y$ on $M$ in $SU_{2,m}/S(U_2U_m)$. 
Then we give another classification in noncompact complex two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$ in terms of the \textit{contact} hypersurface as follows:

\textbf{Theorem 2.6}

Let $M$ be a \textit{contact} real hypersurface in $SU_{2,m}/S(U_2U_m)$ with constant mean curvature. Then one of the following statements holds:

- (B) $M$ is an open part of a tube around a totally geodesic $HH^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$,
- (C) $M$ is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp \bar{J}N$. 
Then we give another classification in noncompact complex two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$ in terms of the \textit{contact} hypersurface as follows:

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The Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$.

In view of the previous results a natural expectation would lead to the totally geodesic $Q^{m-1} \subset Q^m$. Surprisingly, this is not the case. In fact, we will prove

**Theorem 3.1**

Let $M$ be a real hypersurface of the complex quadric $Q^m$, $m \geq 3$. The Reeb flow on $M$ is isometric if and only if $m$ is even, say $m = 2k$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$. 
In $Q^{2k}$ : Berndt and Suh

$M$ : an open part of a tube over $CP^k$ in $Q^{2k}$

$H : H = U(k+1) \hookrightarrow SO(2k+2)$

: Isometry group acting cohomogeneity one
The homogeneous quadratic equation

\[ Q^m = \{ z \in \mathbb{C}^{m+2} | z_1^2 + \ldots + z_{m+2}^2 = 0 \} \subset \mathbb{C}P^{m+1} \]

defines a complex hypersurface in complex projective space \( \mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1) \).

For a unit normal vector \( N \) of \( Q^m \) at a point \( [z] \in Q^m \) we denote by \( A_N \) the shape operator of \( Q^m \) in \( \mathbb{C}P^{m+1} \) with respect to \( N \).

The shape operator is an involution on \( T[z]Q^m \) and \( T[z]Q^m = V(A_N) \oplus JV(A_N) \), where \( V(A_N) \) is the \((+1)\)-eigenspace and \( JV(A_N) \) is the \((-1)\)-eigenspace of \( A_N \).

Geometrically this means that \( A_N \) defines a real structure on the complex vector space \( T[z]Q^m \), or equivalently, is a complex conjugation on \( T[z]Q^m \).
The Riemannian curvature tensor $\tilde{R}$ of $Q^m$ can be expressed as follows:

$$\tilde{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY.$$ 

A nonzero tangent vector $W \in T_z Q^m$ is called singular if it is tangent to more than one maximal flat in $Q^m$.

1. If a conjugation $A \in \mathcal{A}[z]$ such that $W \in V(A)$, then $W$ is singular, that is $\mathcal{A}$-principal.

2. If a conjugation $A \in \mathcal{A}[z]$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then $W$ is said to be $\mathcal{A}$-isotropic.
The Riemannian curvature tensor $\bar{R}$ of $Q^m$ can be expressed as follows:

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Let $M$ be a real hypersurface of $Q^m$ and denote $\xi = -JN$, where $N$ is a (local) unit normal vector field of $M$. For $A \in \mathfrak{u}[z]$ and $X \in T[z]M$ we decompose $AX$ as follows:

$$AX = BX + \rho(X)N$$

where $BX$ is the tangential component of $AX$ and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, AJ\xi) = -g(X, JA\xi) = g(JX, A\xi).$$

Since $JX = \phi X + \eta(X)N$ and $A\xi = B\xi + \rho(\xi)N$ we also have

$$\rho(X) = g(\phi X, B\xi) + \eta(X)\rho(\xi) = g(-\phi B\xi + \rho(\xi)\xi, X).$$

We also define

$$\delta = g(N, AN) = g(JN, JAN) = -g(JN, AJN) = -g(\xi, A\xi).$$
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- Proof of Main Theorem
We assume that $m$ is even, say $m = 2k$. The map
\[
\mathbb{C} P^k \to \mathbb{Q}^{2k} \subset \mathbb{C} P^{2k+1}, \ [Z_1, \ldots, Z_{k+1}] \mapsto [Z_1, \ldots, Z_{k+1}, iZ_1, \ldots, iZ_{k+1}]
\]
gives an embedding of $\mathbb{C} P^k$ into $\mathbb{Q}^{2k}$ as a totally geodesic complex submanifold.

Define a complex structure $j$ on $\mathbb{C}^{2k+2}$ by
\[
j(Z_1, \ldots, Z_{k+1}, Z_{k+2}, \ldots, Z_{2k+2}) = (-Z_{k+2}, \ldots, -Z_{2k+2}, Z_1, \ldots, Z_{k+1}).
\]

Then $j^2 = -I$ and note that $ij = ji$. We can then identify $\mathbb{C}^{2k+2}$ with $\mathbb{C}^{k+1} \oplus j\mathbb{C}^{k+1}$ and get
\[
T_{[z]} \mathbb{C} P^k = \{X + jiX \mid X \in \mathbb{C}^{k+1} \oplus [z]\} = \{X + ijX \mid X \in V(A_z)\}.
\]
The normal space becomes
\[ \nu[z] \mathbb{C}P^k = A\overline{z}(T[z] \mathbb{C}P^k) = \{ X - ijX | X \in V(A\overline{z}) \}. \]

The normal $N$ of $T[z] \mathbb{C}P^k$: $\mathfrak{A}$-isotropic, the four vectors $\{ N, JN, AN, JAN \}$: pairwise orthonormal. The normal Jacobi operator $\bar{R}_N$ is given by
\[
\bar{R}_N Z = \bar{R}(Z, N)N \\
= Z - g(Z, N)N + 3g(Z, JN)JN \\
- g(Z, AN)AN - g(Z, JAN)JAN.
\]

Both $T[z] \mathbb{C}P^k$ and $\nu[z] \mathbb{C}P^k$ are invariant under $R_N$, and $R_N$ has three eigenvalues $0, 1, 4$ according to $R_N \oplus [AN]$, $T[z] \mathbb{Q}^{2k} \ominus ([N] \oplus [AN])$ and $RJN$. 
Normal geodesic in complex quadrics

$M$: an open part of a tube over $\mathbb{C}P^k$ in $Q^{2k}$

Let $\gamma$ be a geodesic in $Q^{2k}$ with $\gamma(0) = [z]$ and $\dot{\gamma}(0) = N$.

Denote $\gamma_{[z]} = T_{[z]} \mathbb{C}P^k \oplus (\nu_{[z]} \mathbb{C}P^k \ominus \mathbb{R}N)$
To calculate the principal curvatures of the tube of radius $0 < r < \pi/2$ around $\mathbb{C}P^k$: the standard Jacobi field method as described in Section 8.2 of Berndt, Console and Olmos.

Let $\gamma$: the geodesic in $Q^{2k}$ with $\gamma(0) = [z]$ and $\dot{\gamma}(0) = N$. $\gamma^\perp$: the parallel subbundle of $TQ^{2k}$ along $\gamma$ defined by

$$
\gamma^\perp(t) = T[\gamma(t)]Q^{2k} \oplus \mathbb{R}\dot{\gamma}(t).
$$

Let us define the $\gamma^\perp$-valued tensor field $R^\perp_\gamma$ along $\gamma$ by $R^\perp_\gamma(t)X = R(X, \dot{\gamma}(t))\dot{\gamma}(t)$. Now consider the $\text{End}(\gamma^\perp)$-valued differential equation

$$
Y'' + R^\perp_\gamma \circ Y = 0.
$$
Let $D$ be the unique solution of this differential equation with initial values

$$D(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad D'(0) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

where the decomposition of the matrices is with respect to

$$\gamma_{[z]} = T_{[z]}\mathbb{C}P^k \oplus (\nu_{[z]}\mathbb{C}P^k \ominus \mathbb{R}N)$$

and $I$ denotes the identity transformation on the corresponding space.

Then the shape operator $S(r)$ of the tube of radius $0 < r < \pi/2$ around $\mathbb{C}P^k$ with respect to $\dot{\gamma}(r)$ is given by

$$S(r) = -D'(r) \circ D^{-1}(r).$$
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If we decompose $\gamma_{[z]}$ further into

$$\gamma_{[z]} = (T_{[z]}\mathbb{C}P^k \ominus [AN]) \oplus [AN] \oplus (\nu_{[z]}\mathbb{C}P^k \ominus [N]) \oplus \mathbb{R}JN,$$

we get by explicit computation that

$$S(r) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \tan(r) & 0 & 0 \\
0 & 0 & -\cot(r) & 0 \\
0 & 0 & 0 & -2\cot(2r)
\end{pmatrix}$$

with respect to that decomposition.
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Proposition 3.1

Let $M$ be the tube of radius $0 < r < \pi/2$ around the totally geodesic $\mathbb{CP}^k$ in $Q^{2k}$. Then the following hold:

1. $M$ is a Hopf hypersurface.
2. The normal bundle of $M$ consists of $\mathfrak{a}$-isotropic singular.
3. $M$ has four distinct constant principal curvatures.

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<thead>
<tr>
<th>principal curvature</th>
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<tr>
<td>$0$</td>
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4. $S\phi = \phi S$.
5. The Reeb flow on $M$ is an isometric flow.
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\begin{array}{|c|c|c|}
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\hline
0 & \mathbb{C} \oplus Q & 2 \\
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2. Isometric Reeb Flow

Hyperbolic Grassmannians

3. Hypersurfaces in $SU_{2,m}/S(U_2 U_m)$
4. Isometric Reeb Flow

Complex Quadrics

5. Real hypersurfaces in $Q^{2k}$
6. Tubes around the totally geodesic $CP^k \subset Q^{2k}$
7. Proof of Main Theorem
Now we investigate real hypersurfaces in $Q^m$ for which the Reeb flow is isometric. From this, we get a complete expression for the covariant derivative as follows:

$$(\nabla_X S)Y = \{d\alpha(X)\eta(Y) + g((\alpha S\phi - S^2\phi)X, Y)$$

$$+\delta\eta(Y)\rho(X) + \delta g(BX, \phi Y) + \eta(BX)\rho(Y}\}\xi$$

$$+\{\eta(Y)\rho(X) + g(BX, \phi Y)\}B\xi + g(BX, Y)\phi B\xi$$

$$-\rho(Y)BX - \eta(Y)\phi X - \eta(BY)\phi BX.$$
From Proposition and Lemma the principal curvature function $\alpha$ is constant. Then we get

$$(\lambda^2 - \alpha \lambda)Y + (\lambda^2 - \alpha \lambda)Z = (S^2 - \alpha S)X = Y.$$ 

By virtue of this equation, we can assert the following propositions:

**Proposition 3.2**

Let $M$ be a real hypersurface in $Q^m$, $m \geq 3$, with isometric Reeb flow. Then the distributions $Q$ and $C \ominus Q = [B\xi]$ are invariant.

**Proposition 3.3**

Let $M$ be a real hypersurface in $Q^m$, $m \geq 3$, with isometric Reeb flow. Then $m$ is even, say $m = 2k$, and the real structure $A$ maps $T_\lambda$ onto $T_\mu$, and vice versa.
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For each point $[z] \in M$ we denote by $\gamma[z]$ the geodesic in $Q^{2k}$
with $\gamma[z](0) = [z]$ and $\dot{\gamma}[z](0) = N[z]$ and by $F$ the smooth map
\[
F : M \longrightarrow Q^m, [z] \longrightarrow \gamma[z](r).
\]
$F$ is the displacement of $M$ at distance $r$ in the direction of $N$.
The differential $d[z]F$ of $F$ at $[z]$ can be computed by
\[
d[z]F(X) = Z_X(r),
\]
where $Z_X$ is the Jacobi vector field along $\gamma[z]$ with $Z_X(0) = X$
and $Z'_X(0) = -SX$. The $\mathfrak{a}$-isotropic $N$ gives that
$R_N = R(Z, N)N$ has the three constant eigenvalues $0, 1, 4$ with
the corresponding eigenbundles
\[
\nu M \oplus (\mathcal{C} \ominus \mathcal{Q}) = \nu M \oplus T_\nu,
\]
\[
\mathcal{Q} = T_\lambda \oplus T_\mu \quad \text{and} \quad \mathcal{F} = T_\alpha.
\]
Rigidity of **totally geodesic submanifolds**: $\implies M$ is an open part of a tube of radius $r$ around a $k$-dimensional connected, complete, **totally geodesic** complex submanifold $P$ of $Q^{2k}$.

Klein classified the **totally geodesic** submanifolds $P$ in $Q^{2k}$ as follows:

The focal submanifold $P$: a **totally geodesic** $Q^k \subset Q^{2k}$ or a **totally geodesic** $\mathbb{C}P^k \subset Q^{2k}$.

$\iff M$ is an open part of a tube around $\mathbb{C}P^k$. 
References

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THANKS FOR YOUR ATTENTION!