Geometry of the visuo-vestibular information

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IMJ (Paris 7)
The labyrinth designs the group of displacement in 3D euclidian space, isomorphic to the linear part of the Galilée’s group: 6 dimensions: 3 parameters of rotation (2 pour the axis and 1 for the angle) detected by the gradient of pressure in the three semi-circular canals, 3 parameters of translation, detected by cristals (otoliths) beyond two (or three) epithelial surfaces.
Let us go back in the past, just before the vertebrates:
An exemplar animal: the tunicate larva, with one eye, one otolith, a chord and muscles
To control movements, to perceive space, is the origin of the brain.
A nice local minimum of motor control (5000000000 years)
Two experimental main points support a galilean structure of vestibular information:

1) In the vestibular cerebellum, neurons have separated data for the gravitation vector and the linear acceleration vector, giving the galilean frame for the head (Yakusheva et al. 2007).

2) Otoliths and canals send informations at the same order with respect to galilean tensors, linear acceleration and rotation velocity, or linear jerk (i.e. third derivative w.r.t. time) and rotation acceleration, that are mainly separated in two systems (that overlap): regular and tonic afferent neurons versus irregular and phasic afferent neurons.
More precisely, the « vestibular space » is the Lie algebra $L(G)$ of six dimensions, made by infinitesimal rotations (velocity) and linear acceleration. This follows from the arguments of Poincaré, telling that « mutual information » between animals and their environment are first coded by the structure of the compensations between voluntary actions and passive activations, giving rize to the euclidian group $G$, then the ambient space itself is given by the indifferences of sensors (visual or tactile), $E=G/H$.

When the same arguments are applied to motions themselves, $G$ must be replaced by the tangent group $T(G)=G\times L(G)$, and the indifference of the vestibular end sensors is measured by $G$ (the set of inertial frames), thus the quotient space is the Lie algebra $L(G)$.

Note an important difference with Physics: in $T(G)$ we find the extension of $G$ by the abelian group $T$ of spatial translations, but not by the group $R$ of temporal translations, confirming that the brain contains no absolute representation of time as a line. Time is active and multiple. But this makes no important changes for the invariance itself.
Manifold of reflexes and sensations: VOR, Nystagmus, VCR, locomotor, postural, respiratory, visceral,...
The CLONS* Project

Artificial vestibular system

Decoding

Learning algorithms

Feedback algorithms

Neurophysiology

Implanted electrodes & electronics

Most of what I present now is joint work with Alain Berthoz, CdF, LPPA

* CLOsed-loop Neural prosthesis for vestibular disorders
Assessing morphology and function of the semicircular duct system: introducing new in-situ visualization and software toolbox

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With the software Ariadne
[14]), there exist three vector field $\vec{N}_n (n=1,2,3)$ harmonics and co-harmonics, i.e. irrotational and divergence free, which are tangent to the surface $V$, such that the circulation of $\vec{N}_n$ in the canal $n$ is $2\pi$ and its circulation in the two other canals is zero. For any force field $\vec{F}$, applied to the head, the three integrals of the scalar product $\vec{N}_n \cdot \vec{F}$ over $W$ represent the fundamental quantities detected by the cupula, they give coordinates for head's angular velocity transmitted to the nerve. Thus the harmonic fields $\vec{N}_n$ are the virtual functional canals. Their orthogonal planes generate a foliation of $W$ by surfaces orthogonal to $V$, and approximatively parallel to the cupula in the ampulla.
Navier-Stokes in the ducts fluid, coupled with Reissner-Mindlin for the elastic cupula

\[
\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla \vec{p} + \mu \Delta(\vec{u}) + \vec{G} - \rho \vec{A} - \rho \frac{d\vec{\Omega}}{dt} \times \vec{x} - \rho \vec{\Omega} \times (\vec{\Omega} \times \vec{x})
\]

\[v_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i),\]
\[\varphi_{ij} = \frac{1}{2}(\partial_i \varphi_j + \partial_j \varphi_i),\]
\[w_i = \frac{1}{2}(\partial_i w - \varphi_i).\]

\[M_{11} = -a \frac{H^3}{12} (\varphi_{11} + \nu \varphi_{22}),\]
\[M_{22} = -a \frac{H^3}{12} (\nu \varphi_{11} + \varphi_{22}),\]
\[M_{12} = M_{21} = -a \frac{H^3}{12} (1 - \nu) \varphi_{12};\]

\[Q_i = \kappa C H w_i; \quad i = 1, 2, 3.\]

\[N_{11} = a H (v_{11} + \nu v_{22}),\]
\[N_{22} = a H (\nu v_{11} + v_{22}),\]
\[N_{12} = N_{21} = a H (1 - \nu) v_{12};\]

Once projected along the Kelvin currents, this gives three approximate o.d.e. s

\[\frac{d^2 Q}{dt^2} \int \frac{\rho ds}{A(s)} + \frac{dQ}{dt} \int \frac{8\pi \mu ds}{A(s)^2} + Q(t) \frac{8\pi \gamma' h}{A_c^2} = -\frac{1}{2\pi} \int_W \Phi \wedge *\alpha.\]

Richard D. Rabbitt

\[\int_W \Phi \wedge *\alpha = \int_D *\alpha \int (\frac{r(s)}{t(s)}) \frac{d\vec{\Omega}}{dt} ds.\]

Note that animals prefer to use an infinite dimensional system for registering a finite dimensional information
Labyrinth information

Crocodylus cataphractus

Motion from the past. A new method to infer vestibular capacities of extinct species

Mouvements anciens. Une nouvelle méthode permettant de déduire les capacités vestibulaires d'espèces éteintes

Romain David\textsuperscript{a,\dagger,}\textsuperscript{,} Jacques Droulez\textsuperscript{b}, Ronan Allain\textsuperscript{a}, Alain Berthoz\textsuperscript{b}, Philippe Janvier\textsuperscript{a}, Daniel Bennequin\textsuperscript{c}

2010. C. R. Palevol. 9 (7-8). P 397-410
Microtomography and 3D Scans
Humans sensibility

A

B

C

Front

Left

Right

Back

Legend:

0%
25%
50%
75%
100%
Set of contacted type I and type II hair cells
At least two types of hair cells and afferent neurons (I,II)
Parallel with vision:
Hair cells correspond to rods and cones (for transduction);
Vestibular afferents to bipolar cells;
Central nuclei to ganglion cells (except vestibular loops);
Then many subcortical areas, thalamus and so on
Discrimination by the labyrinths. The noise and information in the afferent nerve.

Frequencies

\[ f_{n,\text{reg}}(t) \text{ and } f_{n,\text{irreg}}(t) \]

Means

\[ \bar{f}_{n,\text{reg}}(t) \text{ (resp. } \bar{f}_{n,\text{irreg}}(t)) \]

Standard deviations

\[ \sigma_{n,\text{reg}}(t) \text{ (resp. } \sigma_{n,\text{irreg}}(t)) \]

Interspike time interval \( \tau(t) = 1/f(t) \)

\[ CV(\tau) = a(\tau)(CV^*)^{b(\tau)}, \]

CV is the ratio of the standard deviation by the mean interspike, measured experimentally by Goldberg and Fernandez (70, 80’s)
Determination of the mean and the variance of the IFR

\[ E(f(t))E(\ell(t)) \approx 1. \quad \frac{N_k}{k} \to \bar{f}, \quad \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(t)dt = E(f) = \bar{f} \]

\[ E(|f - \bar{f}|^2) = E\left(\frac{1}{\ell} - \frac{1}{\bar{\ell}}\right)^2 = E\left(\frac{|\ell - \bar{\ell}|^2}{\ell^2\bar{\ell}^2}\right) = E\left(\frac{|\ell - \bar{\ell}|^2}{\bar{\ell}^4(1 + \frac{\ell - \bar{\ell}}{\bar{\ell}})^2}\right) \]

\[ |\ell - \bar{\ell}| < \bar{\ell}, \]

\[ E\left(\frac{|\ell - \bar{\ell}|^2}{\bar{\ell}^4(1 + \frac{\ell - \bar{\ell}}{\bar{\ell}})^2}\right) = \sum_{n=0}^{\infty} (-1)^n(n + 1)\bar{\ell}^{-n-4}E(|\ell - \bar{\ell}|^{n+2}) \]

1) Regular neuron

\[ M_{2k} = C_{2k}\sigma(\ell)^{2k}. \]

\[ \sigma^2 \approx \bar{f}^2 \sum_{k=1}^{\infty} (2k - 1)C_{2k}(CV^*)^{2k}. \]

\[ \sigma_{reg} \approx C_{reg}\bar{f}. \]
2) Irregular neuron

\[ \sigma(i) \text{ is proportional to } i^2. \quad |M_{n+2i^{-n-4}}| < (i^n). \]
\[ \sigma(f)^2 \approx M_2(i). \bar{f}^4 = a^2(CV^*)^2 i^4 \bar{f}^4, \]
\[ \sigma_{irreg} \approx C. \]

Both results accord with Sadeghi et al. 2007

More general hypothesis:

\[ \sigma_f \approx C \bar{f}^\beta \quad dP_i(x) = \frac{x^{\lambda-1}e^{-x/\theta}}{\Gamma(\lambda)\theta^\lambda} \, dx. \quad \bar{f} = \frac{1}{\theta(\lambda-1)}, \quad \sigma = \frac{\bar{f}}{\sqrt{\lambda-2}}. \]

3) Cupula deflection

\[ \hat{C}_n(\vec{\Omega})(t) = \int_0^t K_n(t, s) \frac{d\vec{\Omega}(s)}{ds} ds. \]
\[ \hat{C}_n(\vec{\Omega})(t) = \vec{X}_n \cdot \vec{\Omega}(t), \]
\[
\frac{d^2 C_n}{dt^2} + b_n \frac{dC_n}{dt} + c_n C_n = X_n \cdot \frac{d\hat{\Omega}}{dt},
\]
\[
\sigma^2(C_n) \approx \epsilon^2 \hat{C}_n^2 + \eta^2
\]

Linear transfer

\[
\bar{f}_{n,\nu}(t) = \gamma_\nu \hat{C}_n(\hat{\Omega})(t) + \delta_\nu,
\]
\[
f_{n,\nu}(t; \omega, \omega') = \gamma_\nu(t; \omega)C_n(\hat{\Omega})(t; \omega') + \delta_\nu(t; \omega),
\]
\[
Var(f_{n,\nu}) = C^2 |\gamma \hat{C}_n + \delta|^{2\beta} + C^2 |\gamma \sqrt{\epsilon^2 \hat{C}_n^2 + \eta^2} + \delta|^{2\beta} + \gamma^2 (\epsilon^2 \hat{C}_n^2 + \eta^2)
\]
\[
\sigma(f_{n,\nu}) \approx C |\gamma(\hat{C}_n(1 + \epsilon/2) + \eta) + \delta|^{\beta}.
\]
\[
\gamma' = \gamma + \frac{\epsilon \gamma}{2}, \quad \delta' = \delta + \eta \gamma, \quad \bar{f}' = \gamma' \hat{C}_n + \delta'. \quad \sigma_f \approx C |\bar{f}'|^{\beta}.
\]
Information spectrum

\[ R_n(t) = A \overrightarrow{X}_n \cdot \overrightarrow{\Omega}(t) + B. \]

Probability dependent on velocity, i.e. on \( L(G) \)

\[ R'_n(t) = A' \overrightarrow{X}_n \cdot \overrightarrow{\Omega}(t) + B', \]

\[
D_{\text{reg}}(\overrightarrow{\Omega})(x) dx = \frac{e^{-\frac{1}{2} \sum_{1}^{6} (x_n - R_n)^2 / C_{\text{reg}}^2 (R'_n)^{2\alpha}}}{\int_{\mathbb{R}^6} e^{-\frac{1}{2} \sum_{1}^{6} (x_n - R_n)^2 / C_{\text{reg}}^2 (R'_n)^{2\alpha}} dx_1 ... dx_6} \]

Fisher Information metric

\[ g_{ij} = \mathbb{E}_p (\partial_i \ln(p_{\Omega}) \partial_j \ln(p_{\Omega})), \]

\[ g_{ij} = \sum_{1}^{6} (C^{-2} R_n^{-2\alpha} + \frac{(A')^2}{A^2} (\alpha(2\alpha + 1)(R'_n)^{-2} - \alpha(R'_n)^{-\alpha - 1})) \partial_i R_n \partial_j R_n. \]

\[ g_{ij}^0 = C_0^{-2} \sum_{1}^{6} \partial_i R_n \partial_j R_n, \quad \partial_i R_n \partial_j R_n = A^2 X_n^i X_n^j, \]
\[ g_{ij} = \partial_i \partial_j V. \]

Regular case

\[ \alpha = \frac{1}{2} \quad V = \sum_{1}^{6} (C^{-2} (R_n \ln R_n - R_n) - \ln R'_n + 2\sqrt{R'_n}). \]

Irregular case

\[ \alpha = -\frac{1}{2} \quad V = \sum_{1}^{6} \left( \frac{1}{6C^2 R_n^3} + \frac{2}{3} (R'_n)^{3/2} \right). \]

General case

\[ V = \sum_{1}^{6} \left( \frac{R_n^{2-2\alpha}}{C^2 (2 - 2\alpha)(1 - 2\alpha)} + \frac{(R'_n)^{1-\alpha}}{1 - \alpha} - \alpha (1 + 2\alpha \ln R'_n) \right). \]
Discrimination and sensibility

for $\alpha = -1/2$,

$$g(\vec{\delta}, \vec{\delta}') = \sum_{1}^{6} A^2 (X_n \cdot \vec{\delta})^2 (C^{-2} R_n(\vec{\Omega})) + \frac{1}{2 \sqrt{R_n(\vec{\Omega})}}.$$ 

$$\frac{1}{\sqrt{1 + x_n}} + \frac{1}{\sqrt{1 - x_n}} \geq 2 + \frac{3}{4} x_n^2 + ...$$

In conclusion, the decreasing of the sensibility at the level of the cupula and the adaptation of the characteristic noises in the neurons of the vestibular nerve, joined together, permit to augment the discriminability in a wider domain of angular velocity vectors.
A. Li,* J. Xue,* and E. H. Peterson

Architecture of the Mouse Utricle: Macular Organization and Hair Bundle Heights

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Model of the crista (with Prisca Marianelli)

FIGURE 4. Transverse sections: (A) Mouse horizontal crista and (B) macaque superior crista. Central zone is the area between the arrowheads. Scale bar represents 100 μm.
Crista egregia. A three-dimensional mathematical model of the crista ampullaris surface.

Prisca Marianelli · Alain Berthoz · Daniel Bennequin

\[ \sigma^{ij}(x,t) = 2\mu(x)u^{ij}(x,t) + \lambda(x)\text{Tr}(u)g^{ij}(x,t), \quad \text{Tr}(u)(x,t) = \sum_{ij} g_{ij}(x)u^{ij}(x,t), \]

\[ (\nabla_j u)^i = \partial u^i / \partial x^j + \sum_k \Gamma^i_{jk} u^k \quad \sum_i \sigma_{ij}^0 h^i + \sum_i \sigma_{ij}^1 h^i = 0 \quad \beta_i = \sum_j g_{ij} u^j \]

\[ \rho \partial_t^2 \beta + d^* d(\mu \beta + \nu \partial_t \beta) = dp; \quad d^* d(\mu \beta) = dp. \quad d^* \beta = 0 \]

\[ d\mathcal{E} = TdS + \sum_{ij} \sigma^{ij}(x,t)du^{ij}(x,t), \quad \frac{\partial \mathcal{E}_c}{\partial t} = - \int \mu(x,t) \sum_{ijkl} g_{ik} g_{lj} u^{ij} u^{kl} dV(x) \]

\[ \frac{\partial g_{ij}}{\partial x^1} = 0, \quad \frac{\partial b_{ij}}{\partial x^1} = 0 \]

\[ x = B \cos t \cos \theta - A \sin t \sin \theta \]
\[ y = A \theta - B t \]
\[ z = B \cos t \sin \theta + A \sin t \cos \theta, \]

\[ R(y) = a \cosh \left( \frac{y - y_0}{a} \right), \]
\[ u(x,y,z) = \frac{a}{\mu} \frac{R - R(y)}{R} \ln \frac{R}{R(y)} \partial_{\theta}. \]
Crista egregia principles
Minimal surfaces

Catenoids

\[ X(z) = X_1, X_2, X_3 \]
\[ X_1 = -\text{Re}(c(z^{-1} + z)), \]
\[ X_2 = +\text{Im}(c(z^{-1} - z)), \]
\[ X_3 = 2(\cos \theta \ln|z| - \sin \theta \cdot \arg z) \]
Architecture of the Mouse Utricle: Macular Organization and Hair Bundle Heights

A. Li,* J. Xue,* and E. H. Peterson

Maculae and striolae
Striola magica. A functional explanation of otolith geometry

Mariella Dimiccoli · Benoît Girard · Alain Berthoz · Daniel Bennequin

\[ R_i(s; A)(t) = \int duk(t - u)\Theta(\theta(s; u))\Phi(\alpha(s; u)), \]

\[ R_i(s; A)(t) = \Theta(\theta(t))\Phi(\alpha(t)) \]

\[ R_i(s_1, s_2; A)(t) = R_i(s_1; A)(t)R_i(s_2; A)(t). \]

\[ R(a_{ij}, \vec{A}) = R(s_i, \vec{A})R(s_j, \vec{A}) = f_1(\alpha_i)f_2(\beta_i)f_1(\alpha_j)f_2(\beta_j) \]
Optimal information striola

the circular helices

\[ x = a \cos t, \quad y = a \sin t, \quad z = ct. \]

Translation surfaces of Lie and Poincaré

\[ x(s_1, s_2) = f(s_1) + f(s_2), \]
\[ y(s_1, s_2) = g(s_1) + g(s_2), \]
\[ z(s_1, s_2) = h(s_1) + h(s_2). \]

\[ dA = \sqrt{g}\, ds_1\, ds_2, \]
\[ K = \frac{g_{12} \partial_1 (g_{12}) \partial_2 (g_{12}) + g \partial_1 \partial_2 (g_{12})}{g^2} \]
\[ J = \frac{(\vec{t}_1 \cdot \vec{t}_2)(\vec{t}_{11} \cdot \vec{t}_2)(\vec{t}_{22} \cdot \vec{t}_1) + g(\vec{t}_{11} \cdot \vec{t}_{22})}{g^{5/2}} \]
\[ K(s_1, s_2) = -\frac{\tau^2}{4} [1 + (s_1 + s_2 - 2s)(\dot{\tau}/\tau)], \]
Spherical optimal striola

(a natural restriction for the tonic channel)

\[ C(s) = R \cdot \text{cnl}^2\left(\frac{R^{1/2}(s - s_0)}{2A}\right) \quad A = T \sqrt{C} \]

\[ I(u) = \int_u^1 \frac{dt}{\sqrt{1 - t^4}} \]

\[ \text{cnl} \text{ is the inverse of the lemniscate integral:} \]

(M.Dimiccoli, B.Girard, A.Berthoz, D.Bennequin, 2012)
Striola magica, principles
Virtual phasic macula

\[ x(t_1, t_2) = t_1 + t_2, \]
\[ y(t_1, t_2) = t_1^2 + t_2^2, \]
\[ z(t_1, t_2) = t_1^3 + t_2^3. \]
\[ z = \frac{1}{2}x(y + x^2) \]

\[ x = \frac{t}{\sqrt{2}}, \quad y = t^2, \quad z = \frac{\sqrt{2}}{3}t^3 \]
\[ \kappa = \tau = \frac{1}{(1 + t^2)^2} \]

\[ J' = \sqrt{gK} \]

\[ K(s_1, s_2) = -\frac{\tau^2}{4}[1 + (s_1 + s_2 - 2s)(\dot{\tau}/\tau)], \quad g(s_1, s_2) = \kappa^2(s_1 - s_2)^2[1 + (s_1 + s_2 - 2s)(\dot{\kappa}/\kappa)]. \]
A mathematical coincidence?

- For the crista: a minimal surface that is formed by the real points of the translation surface of an isotropic circular helix in C3
- For the sriola, the real translation surface of a real circular helix in R3
- Dualitivity of compact and non-compact symmetric space (Lobatchevski: i versus 1)
- Dualitivity of angular velocities and linear accelerations
Central visuo-vestibular receptive fields

In the brain, space and time are not given first by position and date but they are accessible through impulsion and frequency

\[ R_a(t) = \sum_j \lambda_j u_j \left( \text{Re} \left( \int G_j(x, y, s) I(x, y, t - s) e^{-i(x\xi_0 + y\eta_0 + s\omega_0 + \varphi_j)} \, dx \, dy \, ds \right) \right) \]

\[ = \sum_j \lambda_j u_j \left( \text{Re} \left[ e^{-i\varphi_j} F(G_j I_t)(\xi_0, \eta_0, \omega_0) \right] \right) \]

\[ u_m(x) = C_0 \frac{x^m}{1 + C_\infty x^m}; \]

\[ R_a(t) = \sum_n \sum_{\alpha} u_\alpha [F(I_t \otimes I_t \ldots \otimes I_t \cdot G^m_\alpha)(\xi_{\alpha_1}, \eta_{\alpha_1}, \omega_{\alpha_1}, \ldots, \xi_{\alpha_n}, \eta_{\alpha_n}, \omega_{\alpha_n})] \]
Receptive Fields

\[ R_t(I_{s \leq t}) = \sigma_c [\int dsdxdy e^{i(\xi_0 x + \eta_0 y + \omega_0 s + \varphi_0)} G(x_0 - x, y_0 - y, s - t_0) I(x, y, t - s)] \]

\[ R_t(\mu) = \sigma_c [K_c * (\tau_t \mu)^\vee]. \]

\[ K_c * (\tau_t \mu)^\vee = \int_\mathbb{R} ds \int_G dh K_0(g_0 h^{-1}, s) \langle \psi_1(s) | U^{\lambda(s)}(g_0 h^{-1}) | \psi_2(s) \rangle \mu(h, t - s) \]

\[ R(t) = \int k_0(s) ds \cos(m(\theta - \theta_0)) J_m(\rho_0 \cos(\omega(t - s))) \]

\[ J_t(z) = \sum_0^\infty \frac{(-1)^n (z/2)^{2n+t}}{n! \Gamma(n + t + 1)} \]

\[ R(t) = \rho_0 \int k_0(s) ds \cos(m(\theta - \theta_0)) \cos(\omega(t - s)). \]

\[ R(t) = \rho_0 \omega^{D_0} \cos(m(\theta - \theta_0)) \cos(\omega(t - t')). \]
$$E_a(t) = \langle \chi_0 \tau_{(x_0,y_0,t_0)} \Psi, I_t \rangle = \mathcal{F}(\tau_{(x_0,y_0,t_0)} \Psi, I_t)(\xi_0, \eta_0, \omega_0).$$

With Abelian characters:

$$\chi_0(x, y, t) = e^{i(x \xi_0 + y \eta_0 + t \omega_0)}$$

2D+1 frequency in vision

Or in 3D+1, with disparity:

Akiyuki Anzai$^{1,2}$, Izumi Ohzawa$^{1,3}$ and Ralph D. Freeman$^1$

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B. J. MALONE, V. R. KUMAR, AND D. L. RINGACH

More generally, for many groups, a duality holds between functions on \( G \) and functions on its unitary dual. For instance, for the Galilée group.

\[
\hat{\phi}(\lambda) = \int_G U^\lambda(g) \phi(g) dg \quad \phi(g) = \int_\pi Tr[U^*_\lambda(g) \hat{\phi}(\lambda)] d\pi(\lambda)
\]

Which permits unitary receptive fields for vestibular information:

\[
R_t(\mu) = \sigma_c[K_c \ast (\tau_t \mu)^\vee].
\]

\[
K_c \ast (\tau_t \mu)^\vee = \int_\mathbb{R} ds \int_G dh K_0(g_0 h^{-1}, s) \langle \psi_1(s) | U^\lambda(s)(g_0 h^{-1}) | \psi_2(s) \rangle \mu(h, t - s)
\]

2D simplest case:

\[
R(t) = \int k_0(s) ds \cos(m(\theta - \theta_0)) J_m(\rho_0 \cos(\omega(t - s)))
\]

Approximate:

\[
R(t) = \rho_0 \omega^{D_0} \cos(m(\theta - \theta_0)) \cos(\omega(t - t'_0)).
\]
Galilean Invariant NLRFs

$$\bar{X} = \bar{x} - \bar{v}t \quad ; \quad T = t.$$ 

$$\phi(g) = \int_{\Pi} Tr[U^*_{\lambda}(g)\hat{\phi}(\lambda)]d\pi(\lambda)$$

$$\hat{\phi}(\lambda) = \int_{G} U^\lambda (g)\phi(g)dg$$

$$U^\rho(g)(f)(S) = e^{i\rho \bar{v}.S(\bar{e}_0)}f(R^{-1}S)$$

$$U^\rho(g)(f)(\theta) = e^{i\rho r \cos(\theta - \varphi)}f(\theta - \alpha)$$