Uniform observability of linear time-varying systems and application to robotic problems

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Overview

1. Context
2. Technical result
3. Application
4. Summary and conclusion
Rich literature on observers design:

- **Linear systems**
  \[
  \begin{align*}
  \dot{x} &= A(t)x + B(t)u \\
  y &= C(t)x 
  \end{align*}
  \]
  - Kalman filters/observers
  - Luenberger observers
  - Global convergence properties if observable
  - Optimality properties

- **Nonlinear systems**
  \[
  \begin{align*}
  \dot{x} &= f(x, u, t) \\
  y &= h(x, u, t) 
  \end{align*}
  \]
  - Extended Kalman Filters, Unscented Kalman Filter, Particle Filters
  - High-gains observers
  - Invariant observers
  - Global convergence properties hard to establish
In some cases:

- A nonlinear system can be lifted into a higher-dimensionnal state space
- So as to make the system linear
- This allows one to make use of linear systems theory

In this paper:

- We establish a technical result on observability of Linear Time-Varying Systems (Ph.D. G. Scandaroli)
- And illustrate its application to a robotics estimation problem
Recap on observability of LTV systems:

\[
LTV : \begin{cases}
\dot{x} &= A(t)x + B(t)u \\
y &= C(t)x
\end{cases}
\]

**Assumption:** \( A, B, C \) are continuous and bounded on \([0, +\infty)\).

**Definition:** System LTV is *uniformly observable* if there exist \( \tau, \delta > 0 \) such that

\[
\forall t \geq 0, \quad 0 < \delta I \leq W(t, t + \tau) \triangleq \int_{t}^{t+\tau} \Psi(s, t)^T C^T(s)C(s)\Psi(s, t) \, ds
\]

with \( \Psi(s, t) \) the state transition matrix of \( \dot{x} = A(t)x \) and \( I \) the identity matrix. \( W \) is called *Observability Grammian* of System LTV.
Recap on observability of LTV systems:

**Definition:** (Chen 1984) The Observability Space $\mathcal{O}(t)$ of System LTV is

$$\mathcal{O}(t) \triangleq \left( \begin{array}{c} N_0(t) \\ N_1(t) \\ \vdots \end{array} \right), \quad N_0 \triangleq C, \quad N_{k+1} \triangleq N_k A + \dot{N}_k \quad \text{for } k = 0, \ldots$$

**Remarks:**

- In the time-invariant case, $\mathcal{O}$ is the standard Kalman observability matrix.
- In the time-varying case, $\text{Rank}(\mathcal{O}(t)) = n$ implies instantaneous observability at $t$.
Main technical result:

**Proposition:** Assume that there exists a positive integer $K$ such that:

1. The $k$-th order derivative of $A$ (resp. $C$) is well defined and bounded on $[0, +\infty)$ up to $k = K$ (resp. up to $k = K + 1$).

2. There exist a $n \times n$ matrix $M$ composed of row vectors of $N_0, \ldots, N_K$, and two scalars $\tilde{\delta}, \tilde{\tau} > 0$ such that

$$\forall t \geq 0, \quad 0 < \tilde{\delta} \leq \int_t^{t+\tilde{\tau}} |\det(M(s))| \, ds$$

with $\det(M)$ the determinant of $M$.

Then, System LTV is uniformly observable.

**Remark:** (Bristeau-Petit-Praly, 2010) have provided another sufficient condition, similar to this one, but it requires some positivity at each time-instant $t$, uniform w.r.t. $t$. 
Application

A monocular visuo-inertial problem (Eudes-Morin 2014):

**Estimation Problem:** Estimate $R, p, \dot{p}, n^*, d^*$ from the following measurements

$$y = (H, \omega, a_s), \quad H = R^T - \frac{1}{d^*} R^T p(n^*)^T$$

- $H$ the **Homography Matrix** (computed from visual data)
- $\omega$ the sensor’s angular velocity (measured with IMU’s gyrometers)
- $a_s$ the specific acceleration (measured with IMU’s accelerometers)
Application

A monocular visuo-inertial problem:

- **Solution 1:** (Servant-Houlier-Marchand 2010)

  \[ x = \left( R, p, \dot{p}, n^*, d^* \right), \quad y = (H, \omega, a_s) \]

  and

  \[ \dot{R} = R S(\omega), \quad \ddot{p} = g^* + Ra_s \]

  Dynamics is linear, measurement \( H \) is a nonlinear function of \( x \).

- **Solution 2:** (Mahony-Hamel-Morin-Malis 2012)

  \[ x = (\bar{H}, M, n^*) , \quad y = (\bar{H}, \omega, a_s) \]

  with

  \[ \bar{H} \triangleq \det(H)^{-1/3} H \in SL(3), \quad M \triangleq R^\top \dot{p} n_s^\top, \quad n_s \triangleq \frac{n^*}{d^*} \]

  Dynamics is nonlinear, measurement \( \bar{H} \) is a linear function of \( x \).
A monocular visuo-inertial problem:

- **Solution 3:** (Eudes-Morin 2014)

Lifting the estimation problem in higher dimension: Let

\[ x = (H, M, n_s, Q) \overset{\Delta}{=} R^T g^* n_s^T, \quad y = (H, \omega, a_s) \]

One has:

Lifted:

\[
\begin{align*}
\dot{H} &= -S(\omega)H - M, \\
\dot{M} &= -S(\omega)M + Q + a_s n_s^T, \\
\dot{n}_s &= 0, \\
\dot{Q} &= -S(\omega)Q
\end{align*}
\]

Dynamics is LTV, measurement \( H \) is a linear function of \( x \).
Application

**Proposition:** System Lifted with measurement \( y = (H, \omega, a_s) \) is uniformly observable if

1. \( \omega \) and \( a_s \) are continuous and bounded on \([0, +\infty)\), and their first, second, and third-order time-derivatives are well defined and bounded on \([0, +\infty)\);

2. there exist two scalars \( \delta, \sigma > 0 \) such that

\[
\forall t \geq 0, \quad 0 < \delta \leq \int_{t}^{t+\sigma} \| \dot{a}_s(\tau) + \omega(\tau) \times a_s(\tau) \| \, d\tau 
\]  

(1)

**Remarks:** Condition (1)

- is equivalent to

\[
\forall t \geq 0, \quad 0 < \delta \leq \int_{t}^{t+\sigma} \| \ddot{a}_s(\tau) \| \, d\tau 
\]

- can be used to check in real-time loss of observability and adapt observer gains accordingly.
Summary and Conclusion

- A sufficient condition for uniform observability of LTV systems
- Illustration for a robotic problem via lifting
- Open issue: how far can we go with this lifting approach?

Thank you for your attention!