A Generalized Proportional Integral Output Feedback Controller for the Robust Perturbation Rejection in a Mechanical System

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Abstract—In this article, a Generalized Proportional Integral (GPI) controller is proposed for the efficient rejection of a completely unknown perturbation input in a controlled mass system attached to an uncertain mass-spring-damper mechanical system. We propose a classical compensation network form of the GPI controller, including a sufficient number of extra integrations, which results in a robust perturbation rejection scheme for a trajectory tracking task on the controlled mass subject to the unknown perturbation input. Aside from encouraging simulations, the proposed controller is implemented and tested in a laboratory experimental set up and its robust performance is clearly assessed by using exactly the same controller in three completely different topological situations. The experiments are repeated including infinite dimensional perturbations arising from the effects of several added un-modeled flexible appendages carrying unknown masses.

Index Terms—GPI control, Perturbation Rejection, Flatness.

I. INTRODUCTION

Generalized Proportional Integral (GPI) control was introduced, in the context of Predictive Control of Differentially Flat systems in an article by Fliess and his coworkers [8]. The main idea is to avoid the explicit use of state observers by resorting to structural reconstructions of the state on the basis of iterated integrations of inputs and outputs. The method purposefully ignores initial conditions and classical perturbation inputs (constant perturbations, ramps, quadratic perturbations, etc., i.e., it ignores possibly unstable time-polynomial perturbations). Later on, still at the controller design stage, the method proposes a compensation of the effect of the neglected perturbations and initial conditions -in the state reconstruction task- by adding to the reconstructed state feedback controller a suitable linear combination of iterated output tracking error integrals. GPI controllers for state space described linear systems have been systematically derived in Fliess, Marquez and Delaleau [7]. For time-varying linear systems and some applications in nonlinear Power Electronics we refer the reader to the work of Sira-Ramírez and Silva-Navarro [15] and [16]. In the context of nonlinear systems GPI controllers have been proposed, for the regulation of rigid and flexible robots as well as inertia pendulum wheels, by Hernández and Sira-Ramírez in [10], [11], [12]. In the context of active vibration absorbers, GPI controllers have been previously addressed in the work of Beltrán et al (See [2]). Combinations of GPI controllers and fast algebraic identification techniques for adaptive attenuation of vibrations may be found in Beltrán et al. [3], and in [4]. GPI controllers merge quite well with sliding mode control techniques making it unnecessary to know the full state of the system, as evidenced in Beltrán et al. [5]. GPI controllers have been used in several experimental set ups of power electronics problems in the works of Sira-Ramírez and Silva-Ortigoza [14] and more recently, for multivariable cases, in the work Franco et al. [1]. GPI controllers have been found to be expressible in proper transfer function form, or in classical compensation network form, in an article by Becedas et al. [1], where extensive experimental tests were successfully carried out, for controlling highly flexible manipulators using a combination of GPI controllers and algebraic identification techniques as advocated in Fliess and Sira-Ramírez in [6]. For other developments concerning GPI control of flexible structures, see also Trapero et al. [17]. It is in this latter form, of classical compensation networks, that the greatest advantages of GPI control techniques emerge as related to controller simplicity, ease of implementation, reliability and numerical precision.

In this article, we consider a mass, capable of sliding along horizontal guidelines, and controlled by an external input force. The mass is attached to a train of similar, but unknown, sliding masses joined in cascade by springs of unknown values. The number of masses affecting the motions of the directly controlled mass is completely unknown as it is unknown whether or not the very first and the very last masses of the train system are free or are they attached to some fixed point. We propose an output feedback controller of the GPI type, for a reference trajectory tracking task, which is based on position measurements of the controlled mass alone. The proposed controller is robust with respect to the perturbation force acting on the first controlled mass as a result of the vibrations of the rest of the uncertain cascaded mass-spring-damper system and the spring and damper possibly attaching the mass to a fixed point. In order to assess the performance and outstanding disturbance rejection capabilities of the proposed robust GPI controller, we also carry out the same previous experiments including one, two and three, vertical flexible appendages attached to the moving masses with a mass placed at the free end of the appendage.

Our GPI control design rationale is as follows: We first design an output feedback trajectory tracking controller of the GPI type for the controlled mass which is made robust
with respect to an entire family of $p - 1$-th order unknown polynomial perturbation inputs. Clearly, any representative of such a polynomial input family is unbounded as long as its order is higher or equal than 1. Thanks to the internal model principle, the resulting output feedback controller, will exhibit $p$ integrators and, in order to overcome the unstable perturbation input it has to, itself, exhibit an unstable behavior while being successful at quenching the tracking error asymptotically exponentially to zero. We next ask ourselves the following question: What is the nature of the closed loop response, and of the controller behavior, when we use the previously designed (potentially unstable) controller on a perturbed system whose perturbation input is no longer an unstable time polynomial signal but a bounded, sufficiently smooth, possibly state-dependent input signal? Since the controller is now facing a bounded perturbation which only locally looks like the previously hypothesized polynomial input, the resulting controller must be still successful in producing a bounded and exponentially stable tracking error response which is close to zero but, also, the controller output will no longer be an unbounded signal but a bounded one. This is due to the fact that the closed loops system is made into a linear exponentially asymptotically stable system processing a bounded residual signal representing the small difference between the actual bounded input and the hypothesized local polynomial approximation at the end of the “moving window” of validity of the polynomial approximation.

Section 2 formulates the problem treated in this article. In this section we solve the closely related problem of designing an output feedback controller, for a single controlled mass, which asymptotically rejects an unstable time-polynomial additive perturbation input while tracking a given output reference trajectory. In section 2 we also justify the boundedness of the closed loop performance of the previously designed output feedback controller when the additive perturbation input is no longer a time-polynomial signal but a bounded, sufficiently smooth, perturbation input signal. Section 3 presents a direct application of the above result to the problem of robustly regulating a sliding mass perturbed by an unknown interaction with an uncertain number of sliding masses attached to the regulated mass in a cascade, tandem, connection using springs with unknown coefficients. Each unknown connected mass will also exhibit a damper modeling the viscous friction existing over the guidelines. The simulation results, presented in this section, motivated the testing of the control scheme on an experimental laboratory set up. We describe and report the successful experimental results in Section 4 where we also include experiments with unknown vibrating flexible attachments. Section 5 contains the conclusions and suggestions for further research.

II. Problem Formulation

Consider the mass-spring-damper system shown in Figure 1.

Suppose this “train” of masses, springs and dampers is constituted, after the first, actuated, known mass $m$, by an uncertain number of connected masses. In other words, we do not know how many masses are there in the train attached to the first mass, nor do we know whether or not the last mass in the train is attached to a wall and, moreover, we do not even know the values of the masses, springs stiffness coefficients and dampers viscous friction coefficients connecting any two consecutive masses beyond the first one. Notice that such an unknown perturbation is, strictly speaking, a state dependent perturbation since the position and the velocity of the first mass contribute to conform the unknown perturbation input. In spite of this, considering the dynamic perturbation as an unknown time signal will produce a successful and quite robust GPI controller.

It is desired to obtain a feedback controller, based only on the measurement of the position variable, $x$, of the directly actuated mass, so that this first mass position is accurately controlled to follow a given reference trajectory $x^*(t)$, in spite of the nature of the uncertain subsystem, attached through a spring and a damper to this first mass, and acting as a perturbation input force of completely unknown nature.

We reduce the posed problem to the following one: Given the following perturbed second order system:

$$m\ddot{x} = u + \xi(t)$$

where $m$ is the known value of the mass, $u$ is the externally applied control input force, the signal $\xi(t)$ is the total, bounded and sufficiently smooth, perturbation force exercised by the motions of the uncertain train system affecting the controlled motions of the single mass system, devise a robust output feedback controller for $u$ which causes the mass position, $x$, to track a given reference position trajectory $x^*(t)$, within a possible tracking error, $x - x^*(t)$, taking values in a small neighborhood of zero, in spite of the perturbation input values.

A. An introductory related problem

Consider the following related problem: Given a second order perturbed system

$$m\ddot{x} = u + \xi(t)$$

control $x$ to track a given smooth reference trajectory $x^*(t)$ irrespectively of the coefficient values of the finite $p - 1$-th order time polynomial signal $\xi(t)$ i.e.,

$$\xi(t) = \sum_{i=0}^{p-1} \gamma_i t^i.$$
Although this is quite an unrealistic problem, given that any polynomial signal necessarily grows without bound as $t$ grows, it will require of a feedback controller action whose output signal, $u$, will also grow in a polynomial fashion. Nevertheless, and in spite of this unstable behavior, let us proceed to synthesize a controller for which the tracking error $e = x - x^*(t)$ exponentially asymptotically decreases to zero, even if at the costly expense of the internal feedback controller instability. We proceed as follows:

Consider the nominal unperturbed system:
\[ m\ddot{x}^*(t) = u^*(t) \]
the tracking error $e = x - x^*(t)$ evolves according to the perturbed dynamics
\[ m\ddot{e} = e_u + \xi(t) \]
with $e_u = u - u^*(t)$. Since $\xi(t)$ is a $p-1$-th order polynomial, its annihilator is constituted by the differential operator $s^p$. We obtain, after taking $p$ time derivatives on the tracking error system, the following expression for the evolution of $e$:
\[ me^{(p+2)}(t) = e^{(p)}(t) = e_v(t) \]
where $e_v(t) = v - v^*(t)$ is an auxiliary control input error representing the $p$-th order derivative of the original control input $u$. Notice that, $v^*(t) = m|\dot{x}^*(t)|^{(p+2)}$. An output feedback controller for $v$ which guarantees asymptotic exponential stabilization of the tracking error $e$ to zero is given by the following GPI controller, written in classical compensation network form, where, in its expression, we have abusively combined time domain quantities with frequency domain quantities, as it is customary in many areas of modern automatic control:
\[ v = v^*(t) - m \left[ k_{p+1}s^{p+1} + k_ps^p + \cdots + k_1s + k_0 \right] e(t) \]
The closed loop tracking error system is seen to evolve, after some algebraic manipulations, as governed by the following symbolic representation of a $2p+3$-th homogeneous ordinary differential equation for the tracking error $e = x - x^*(t)$:
\[ \left( s^{2p+3} + k_{p+2}s^{2p+2} + \cdots + k_1s + k_0 \right) e = 0 \]
Clearly, the proper choice of the control network coefficients $\{k_{p+2}, \cdots , k_1, k_0\}$, representing also the coefficients of the closed loop characteristic polynomial, as those of a Hurwitz polynomial, results in a globally exponentially asymptotically stable equilibrium point, $e = 0$, at the origin of the tracking error system space.

The controller for the original input $u$ is, according to the relation between $u$ and the auxiliary control input $v$, given by:
\[ u = u^*(t) - mG(s)(x - x^*(t)) \]
with
\[ G(s) = \left[ \frac{k_{p+1}s^{p+1} + k_ps^p + \cdots + k_1s + k_0}{s^p(s^{p+1} + k_{p+2}s^{p+2} + \cdots + k_{p+3}s + k_{p+2})} \right] \]

The following elementary magnitude check on the closed loop controlled system
\[ me^{(2)} = e_u + \xi(t) \]
leads us to conclude that the controller output $e_u$ is to be regarded as an unbounded signal. Indeed, since $e$ exponentially asymptotically decreases to zero, while $\xi(t)$, due to its polynomial character, either grows to plus infinity (when $p-1$ is even) or to minus infinity (whenver $p-1$ is odd), then, necessarily $e_u$ will counteract the growing perturbation at the expense of its own stability. i.e., $e_u$, respectively, grows to either minus infinity or to plus infinity so that the signal $e$, and its time derivatives, remain bounded and decreasing to zero.

**B. Some natural questions and their answers**

The natural questions to be asked are the following: What happens to the tracking error $e$, and to the controller behavior, when the perturbation input signal, $\xi(t)$, is **no longer** an unbounded polynomial input but a **bounded, sufficiently smooth**, input signal, and we still insist in using the previously designed output feedback controller $G(s)$? Will the controller output $e_u$ still behave in an unstable fashion or will it result in a bounded control input error?.

The answer to these questions are: The designed output tracking feedback controller, which is ready to sustain asymptotic stability of the controlled system when subject to an unstable input, will be now processing a bounded perturbation input which will not cause the control input to grow in a polynomial fashion. We have “fooled” the controller making it ready to control a polynomial perturbed system while, actually, the foreseen unstable perturbation never shows up. Moreover, since the actual bounded smooth perturbation, locally in time, looks like any one of the representatives of the polynomial family with respect to which the controller has been made to act robustly, the controller effect is to try to regulate to zero, at every moment, the tracking error. Since the actual perturbation differs only slightly from one of the members of the annihilated polynomial family, the closed loop system is only affected by the residual perturbation input comprising the difference between the assumed polynomial input and the actual smooth, bounded, perturbation input. The net result is that the closed loop system, which has been made to be an asymptotically exponentially stable linear system, is in fact **locally** perturbed by a small bounded perturbation signal which is continuously obtained from the residual difference signal between a bounded signal and its locally approximating polynomial. The tracking of the given trajectory will be accomplished in an approximate manner with a tracking error signal uniformly living inside a small radius ball centered at the origin of the tracking error state space.

**C. Some motivating simulations**

In order to graphically evaluate our previous reasoning, suppose we would like to stabilize to, say, zero the output, $y = x$, of the second order perturbed system: $m\ddot{x} = u + \xi(t)$ with $\xi(t)$ being a polynomial input of, say, fourth degree. We thus have
\(x^*(t) = 0\) for all \(t\) and correspondingly, \(u^*(t) = 0\). Clearly, five time derivatives of the system expression annihilate the polynomial perturbation input, \(x(t)\). We use the following GPI controller

\[
u = -m \left[ \frac{k_6 s^6 + k_5 s^5 + \cdots + k_1 s + k_0}{s^6 + k_{12} s^5 + \cdots + k_8 s + k_7} \right] x
\]

The closed loop characteristic polynomial is simply given by

\[p(s) = s^3 + k_{12} s^2 + \cdots + k_1 s + k_0\]

Equating term by term, the coefficients of the closed loop characteristic polynomial with the corresponding ones of the following Hurwitz polynomial:

\[p_d(s) = (s^2 + 2 \zeta \omega_n s + \omega_n^2)^6(s + p)\]

we obtain the required controller gains \(\{k_{12}, k_{11}, \cdots, k_1, k_0\}\) as follows:

\[
\begin{align*}
  k_{12} &= p + 12 \zeta \omega_n \\
  k_{11} &= 12 \zeta \omega_n p + 60 \zeta^2 \omega_n^2 + 6 \omega_n^2 \\
  k_{10} &= 6 \omega_n^2 p + 160 \zeta^3 \omega_n^3 + 60 \zeta^2 \omega_n^2 p + 60 \zeta \omega_n^3 \\
  k_9 &= 240 \zeta^2 \omega_n^4 + 240 \zeta^4 \omega_n^4 + 15 \zeta \omega_n^4 p + 60 \zeta \omega_n^3 p \\
  k_8 &= 12 \zeta \omega_n^5 + 192 \zeta^5 \omega_n^5 + 240 \zeta^4 \omega_n^4 p + 480 \zeta^3 \omega_n^3 p + 15 \zeta \omega_n^4 p + 240 \zeta^2 \omega_n^2 p \\
  k_7 &= 360 \zeta^5 \omega_n^6 + 480 \zeta^4 \omega_n^6 + 120 \zeta \omega_n^5 p + 64 \zeta^4 \omega_n^5 + 192 \zeta^3 \omega_n^5 p + 20 \zeta \omega_n^4 p + 480 \zeta^3 \omega_n^3 p \\
  k_6 &= 20 \zeta \omega_n^5 + 64 \zeta^4 \omega_n^5 p + 120 \zeta \omega_n^7 + 480 \zeta^4 \omega_n^4 p + 360 \zeta^2 \omega_n^2 p + 480 \zeta^3 \omega_n^3 + 192 \zeta^5 \omega_n^5 \\
  k_5 &= 192 \zeta \omega_n^7 p + 480 \zeta^5 \omega_n^7 p + 120 \zeta \omega_n^7 p + 240 \zeta \omega_n^8 + 240 \zeta \omega_n^8 p + 15 \zeta \omega_n^7 p \\
  k_4 &= 160 \zeta^3 \omega_n^8 + 60 \zeta \omega_n^9 + 15 \zeta \omega_n^8 p + 240 \zeta^4 \omega_n^8 p + 240 \zeta^2 \omega_n^8 p \\
  k_3 &= 60 \zeta^2 \omega_n^{10} + 60 \zeta \omega_n^{10} + 6 \omega_n^{10} + 160 \zeta^3 \omega_n^9 p \\
  k_2 &= 6 \omega_n^{10} p + 12 \zeta \omega_n^{11} + 60 \zeta^2 \omega_n^{10} p \\
  k_1 &= \omega_n^{12} + 12 \zeta \omega_n^{11} p \\
  k_0 &= \omega_n^{12} p
\end{align*}
\]

For the simulation results shown below we used the following data for the plant and the controller

\[
m = 1, \quad \zeta = 4, \quad \omega_n = 20, \quad p = 20,
\]

The polynomial input was set to be a fourth degree polynomial given by:

\[
\xi(t) = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 + \gamma_4 t^4
\]

with \(\gamma_0 = 1.0 \times 10^{-3}\) for all \(i^1\).

Clearly, as the polynomial perturbation \(\xi(t)\) grows, the control input \(u\) grows in the opposite direction canceling out the perturbation effect on the closed loop controlled system. The output signal \(x\) is seen to converge to zero in an asymptotically exponential fashion.

Fig. 2. Closed loop response of polynomial perturbed second order system clearly exhibiting the canceling effect of the unstable control input.

The following figure depicts the performance of exactly the same feedback controller designed above when the perturbation signal \(\xi(t)\) is given by a sufficiently smooth but bounded signal. We used the following perturbation input in the simulations

\[
\xi(t) = \gamma \sin(t) \exp(-\sin^2(3t)), \quad \gamma = 1.0 \times 10^{-3}
\]

Fig. 3. Closed loop response of bounded, non-polynomial perturbed second order system depicting the canceling effect of the bounded control input.

The output signal \(x\) converges now to the interior of a small radius ball centered around the origin of the error space, while
the controller output $u$ remains bounded only approximately canceling the effects of the smooth, bounded, perturbation input $\xi(t)$. The approximate opposite nature of the control input with respect to the bounded perturbation input is clearly due to the simple magnitude balance in the system dynamics:

$$m\ddot{x} = u + \xi(t).$$

Indeed, now $x(t)$ converges to a small vicinity around zero, and its first and second order time derivatives remain bounded. It follows that, necessarily, $u$ and $\xi(t)$, nearly “point-wise” cancel each other in the sum, $u + \xi(t)$, on the right hand side of the system dynamics.

### D. Main Result

**Theorem 1**: Consider the perturbed system:

$$m\ddot{x} = u + \xi(t)$$

with $m$ being the known mass, $u$ is the externally applied force acting as a control input and let $\xi(t)$ be an uniformly bounded, sufficiently smooth signal such that, given any integer $p > 1$, $\xi(t)$ can be written as follows:

$$\xi(t) = \xi_0 + \xi_1 t + \frac{1}{2!}\xi_2 t^2 + \cdots + \frac{1}{(p-1)!}\xi_{p-1} t^{p-1} + o_p(t)$$

with

$$\xi_i = \frac{d^i\xi(t)}{dt^i} \bigg|_{t=0}, \quad i = 0, 1, \ldots, p-1$$

and $|o_p(t)|$ is a bounded signal uniformly contained within a disk of small radius $\epsilon_p$ centered at the origin with $o_p(0) = \delta_p(0) = \cdots = 0$. Given a smooth, desired, reference trajectory, $x^*(t)$, for the controlled position $x$, the robust GPI feedback controller including $p$ extra integrations:

$$u = u^*(t) - m \left[ \frac{k_{p+1} s^{p+1} + \cdots + k_1 s + k_0}{s^p (s^{p+1} + k_{p+2} s^{p+2} + \cdots + k_{p+2} s^{p+2})} \right] (x - x^*(t))$$

with $u^*(t) = m\ddot{x}^*(t)$ being the nominal input of the second order mass system, globally asymptotically exponentially stabilizes the position tracking error $e = x - x^*(t)$, for the closed loop perturbed system, towards the interior of a small neighborhood of zero, provided the set of coefficients \( \{k_{p+2}, k_{p+1}, \ldots, k_1, k_0\} \) of the output GPI feedback tracking controller are chosen so that the closed loop characteristic polynomial, $p_{cl}(s)$, is a Hurwitz polynomial given by,

$$p_{cl}(s) = s^{2p+3} + k_{p+2} s^{2p+2} + \cdots + k_{p+2} s^{p+2} + k_{p+1} s^{p+1} + \cdots + k_1 s + k_0$$

**Proof**

Let $e(t) = x - x^*(t)$ and let $p_{cl}(\frac{d}{dt})$ denote the differential polynomial obtained from $p_{cl}(s)$ after letting $s = \frac{d}{dt}$. Consider the closed loop system in the time domain. We have after some elementary manipulations:

$$p_{cl}(\frac{d}{dt}) e(t) = \sum_{j=0}^{p+1} k_{p+3-j} \frac{d^{p+1-j}}{dt^{p+1-j}} o_p(t)$$

with $k_{p+3} = 1$. Clearly, under the hypothesis of a small residual, $o_p(t)$, in the vicinity of the current time $t$, the above equation represents an asymptotically exponentially stable system perturbed by a smooth, uniformly bounded signal of very small amplitude. By well known results, (See, for instance the excellent book by Rugh [13]) the closed loop tracking error response trajectory, $e(t)$, remains uniformly bounded by a small radius disk centered around the origin. The robustness result follows.

### III. Simulation Results

We simulated the following mechanical system which coincided with the topology of the experimental set up.

\[
\begin{align*}
    m_1 \ddot{x}_1 &= u + \xi(t) \\
    \xi(t) &= -k_1 x_1 - c_1 \dot{x}_1 - k_2 (x_1 - x_2) \\
    m_2 \ddot{x}_2 &= -k_2 (x_2 - x_1) - c_2 \dot{x}_2 - k_3 (x_2 - x_3) \\
    m_3 \ddot{x}_3 &= -k_3 (x_3 - x_2) - c_3 \dot{x}_3
\end{align*}
\]

The unknown state-dependent perturbation input $\xi(t)$ was modeled as a fourth order time polynomial. A trajectory tracking task was adopted to have the controlled mass position $x$ track, a Bézier polynomial smoothly interpolating between zero and a final position located at one centimeter from the initial rest position in approximately 3 [sec]. The controller was set to be:

$$u = u^*(t) - m \left[ \frac{k_0 s^6 + k_5 s^5 + \cdots + k_1 s + k_0}{s^5 (s^6 + k_1 s^5 + \cdots + k_3 s + k_7)} \right] (x - x^*(t))$$

The controller gains $\{k_1, k_2, \ldots, k_7\}$ were set to coincide with those of the desired characteristic polynomial $(s^2 + 2 \omega_n s + \omega_n^2)^6$ with

$$\zeta = 7, \quad \omega_n = 60, \quad p = 60.$$

The various coefficients were set to be approximately those of the experimental system:

\[
\begin{align*}
    k_1 &= 525.25 \ [N/m], \quad m_1 = 2.78 \ [Kg], \\
    c_1 &= 3.8122 \ [N - s/m], \\
    k_2 &= 328.34 \ [N/m], \quad m_2 = 2.56 \ [Kg], \\
    c_2 &= 1.11 \ [N - s/m], \\
    k_3 &= 185.69 \ [N/m], \quad m_3 = 2.56 \ [Kg], \\
    c_3 &= 1.11 \ [N - s/m].
\end{align*}
\]

### IV. Experimental Results

**A. Perturbation input arising from a finite dimensional system**

The following picture depicts the experimental ECP® system on which we tested the proposed output feedback tracking control strategy of the GPI type with disturbance rejection features based on iterated integrations arising from a polynomial perturbation model annihilation via repeated differentiations of the simplified input output relation.

\[\text{\footnote{Clearly, one can take the bounded time perturbation to be state dependent, in this case, because we know that the neglected contribution of the state function, considered as part of the unknown perturbation, still produces a stable system. Unknown expressions of the state which cause unstable open loop behavior must, in general, be known and exactly canceled on line.}}\]
We first run the tracking controller when no masses were attached to the first controlled mass. Figure 6 shows the obtained closed loop response. We then run the experiment again using exactly the same controller as before, but, this time, with only one (unknown) mass attached to the controlled system. The corresponding closed loop responses are shown in Figure 7. Finally, we run a third experiment using exactly the same previous controller, but, this time, placing two unknown masses in tandem attached to the first controlled mass. The results are shown in Figure 8.

B. Perturbation inputs arising from an infinite dimensional system

In view of the successful experimental performance of the closed loop finite dimensional perturbed mechanical system and the proposed robust GPI controller, we set out to repeat some of the previous experiments with an added complexity in the unknown perturbation signal generation. A rather flexible appendage, topped with a significant mass, was built and firmly attached to the top of one of the moving cars in the train as shown in the following picture.

The experimental results regarding the controlled mass
position and the applied control input force are depicted in Figure 10.

Since the results were rather encouraging, we placed one such flexible appendage on the second and third carts and proceeded to control the first mass motion, as done before, with the same previously used controller. Figure 11 depicts the arrangement. Figure 12 depicts the corresponding controlled position and control input signals.

Finally, we placed a flexible appendage on each one of the three cars in the ECP® system, as shown in the picture of Figure 13. The closed loop performance is depicted in Figure 14. For these experiments the values of the springs, dampers and cart masses were modified to allow for further oscillations. We set:

\[
\begin{align*}
k_1 &= 191.31 \text{ N/m, } c_1 = 3.64 \text{ N/m/s, } m = 2.82 \text{ kg} \\
k_2 &= 391.16 \text{ N/m, } c_2 = 1.75 \text{ N/m/s, } m_2 = 2.59 \text{ kg} \\
k_3 &= 344.83 \text{ N/m, } c_3 = 1.75 \text{ N/m/s, } m_3 = 2.59 \text{ kg}
\end{align*}
\]

The controller gains were set to be

\[
\zeta = 7, \quad \omega_n = 70, \quad p = 70.
\]

C. Robustness with respect to mass uncertainty

As stated in the proposed robust output feedback GPI controller design, given by

\[
u = u^*(t) - m \left[ \frac{k_6 s^6 + k_5 s^5 + \cdots + k_1 s + k_0}{s^6 + k_12 s^5 + \cdots + k_8 s + k_7} \right] (x - x^*(t))
\]

the only parameter that needs to be known, with certitude, is the controlled mass parameter, \( m \), associated with the first cart. Clearly, it is possible that such a mass is not perfectly known. We carried out an experimental study on the effects of
Oscillatory is achieved. The response trajectories for each value of the ratio $m$ controlled responses are less oscillatory while the control crosses the imaginary axis towards the right half of the complex plane. There is therefore an interval of instability caused by the small values of the ratio $m_c/m$. As the ratio grows, the root locus intersects back the imaginary axis and converges towards the stable location of the closed loop zeroes reaching them when $m_c/m = 1$. As the ratio $m_c/m$ approaches “infinity” the roots of the closed loop characteristic polynomial asymptotically tend towards the stable zeros at infinity making the system more stable, faster, and hence, more sensitive with respect to high frequency measurement noises. In conclusion, these results demonstrate that the proposed output feedback control method is robust with respect to a wide range of uncertainty on the only parameter needed to be specified in the controller.

V. CONCLUSION

In this article we have proposed a dynamical output feedback control, of the GPI type, for a reference trajectory tracking task of the position of a controlled mass attached to an unknown dynamical system of the mass-spring-dampers type. The controller is derived on the basis of a severely simplified mechanical system model contemplating only the controlled mass and a bounded, sufficiently smooth, perturbation input but otherwise of completely unknown nature. The actual controlled mass is subject to unknown but bounded perturbation inputs as arising from an uncertain dynamical subsystem of an arbitrary, unknown, number of tandem connected sliding masses via springs and dampers which is attached to the controlled system through a set of springs and dampers with unknown rigidity and viscosity coefficients. The attachment of flexible appendages, actually constituting infinite dimensional systems contributing to the unknown perturbation input, does not change the perturbation rejection capabilities of the proposed output feedback control scheme while satisfactorily accomplishing the desired trajectory tracking control objective.

The developed robust GPI controller is derived on the basis of the simplified second order system controlled mass model and the assumption of a polynomial family of perturbation inputs of certain finite, but arbitrary, degree. The GPI controller adopts the form of a classical scalar compensation network processing the tracking error and producing the required feedback control input force. The performance of the designed controller was tested on a didactic ECP® system using a Matlab Simulink programming facility. The experimental results indicate a remarkable performance of the designed feedback controller in the presence of a trajectory tracking task and completely unknown perturbations which include real Coulomb friction coefficients. An experimental study of the robustness with respect to the resulting controller gain parameter when the mass of the controlled cart is uncertain was convincingly carried out with the presence of the flexible appendages mounted on each one of the carts comprising the ECP® system.

A similar GPI controller, which now controls the position of the second, intermediate, mass in a set of three coupled masses, with the inclusion of a flexible appendage (i.e., an infinite dimensional system), firmly attached to the third mass, is now being pursued as a crucial test on the perturbation...
rejection capabilities of the class of proposed output feedback controllers.

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