Riemannian metrics on Shape spaces of curves and surfaces

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Part I

1. What are the Model spaces of infinite-dimensional geometry?
2. What are the Tools from Functional Analysis?
3. What are the Toys we can play with?
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Part II

A trap of infinite-dimensional geometry: there exist Poisson brackets that are not given by bivector fields

Part III

1. Shape spaces as Quotient versus Sections of fiber bundles
2. Riemannian metrics on Shape spaces
3. Gauge invariant metrics on Shape spaces
What are the Model spaces of infinite-dimensional geometry?

**Hilbert**, **Banach**, **Fréchet**, **Locally Convex spaces**

**Hilbert space** $H = \text{complete vector space for the distance given by an inner product} = \langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}^+$

- **symmetric**: $\langle x, y \rangle = \langle y, x \rangle$
- **bilinear**: $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$
- **non-negative**: $\langle x, x \rangle \geq 0$
- **definite**: $\langle x, x \rangle = 0 \Rightarrow x = 0$

$H^* = H$ (Riesz Theorem).
What are the Model spaces of infinite-dimensional geometry?

Hilbert \( \subset \) **Banach** \( \subset \) Fréchet \( \subset \) Locally Convex spaces

**Banach space** \( B = \) complete vector space for the distance given by a norm \( = || \cdot || : B \rightarrow \mathbb{R}^+ \)

- triangle inequality: \( ||x + y|| \leq ||x|| + ||y|| \)
- absolute homogeneity: \( ||\lambda x|| = |\lambda||x|| \)
- non-negative: \( ||x|| \geq 0 \)
- definite: \( ||x|| = 0 \Rightarrow x = 0 \).

\( B^* = \) Banach space.
What are the Model spaces of infinite-dimensional geometry?

Hilbert $\subset$ Banach $\subset$ Fréchet $\subset$ Locally Convex spaces

**Fréchet space** $F = \text{complete Hausdorff vector space for the distance } d : F \times F \to \mathbb{R}^+$ given by a countable family of semi-norms $\| \cdot \|_n :$

$$d(x, y) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

$F^* \neq \text{Fréchet space in general, but locally convex}$

$F^{**} = \text{Fréchet space.}$
What are the Model spaces of infinite-dimensional geometry?

Hilbert ⊂ Banach ⊂ Fréchet ⊂ **Locally Convex spaces**

**Locally Convex spaces** = Hausdorff topological vector space whose topology is given by a (possibly not countable) family of semi-norms.

**References:**
- Klingenberg : *Riemannian Geometry*
- Lang : *Differential and Riemannian manifolds*  
  *Fundamentals of Differential Geometry*
- Hamilton : *The inverse function theorem of Nash-Moser*
- A. Kriegl and P. Michor : *Convenient setting of Global Analysis*
What are the Tools from Functional Analysis?

Banach-Picard fixed point Theorem

\[(E, d) \text{ complete metric space} \]
\[f : E \to E \text{ contraction of } E : d(f(x), f(y)) \leq kd(x, y) \text{ where } k \in (0, 1)\]

\[\Rightarrow \begin{cases} 
\exists ! x \in E, f(x) = x \\
\forall x_0 \in E, \text{ the sequence } x_{n+1} = f(x_n) \text{ converges to } x
\end{cases}\]
What are the Tools from Functional Analysis?

**Hahn-Banach Theorem**

$E$ locally convex space
$A \subset E$ a convex
$x \in E$, $x \notin \overline{A}$

$\Rightarrow \exists$ continuous functional $\ell : E \to \mathbb{R}$ with $\ell(x) \notin \overline{\ell(A)}$
What are the Tools from Functional Analysis?

Open mapping Theorem

\[
\begin{cases}
  F \text{ Fréchet} & \text{or} & F \text{ webbed locally convex} \\
  G \text{ Fréchet} & \text{or} & G \text{ inductive limit of Baire locally convex spaces}
\end{cases}
\]

\[L : F \to G \text{ continuous, linear, and surjective} \Rightarrow L \text{ is open}\]
What are the Tools from Functional Analysis?

**Inverse function Theorem**

Let $f : \mathcal{U} \subset B_1 \to B_2$ be a $C^1$-map between Banach spaces. If $Df(a)$ is invertible at $a \in \mathcal{U}$, then there exists an open neighborhood $\mathcal{V}_a$ of $a \in \mathcal{U}$ and an open neighborhood $\mathcal{V}_{f(a)} \subset B_2$ such that $f : \mathcal{V}_a \to \mathcal{V}_{f(a)}$ is a $C^1$-diffeomorphism.

**Counterexample** : $\exp : \text{Lie}(\text{Diff}(S^1)) \to \text{Diff}(S^1)$ not locally onto.

**Theorem (Nash-Moser)**

Let $f : \mathcal{U} \subset F_1 \to F_2$ be a smooth tame map between Fréchet spaces. Suppose that the equation for the derivative $Df(x)(h) = k$ has a unique solution $h = L(x)k$ for all $x \in \mathcal{U}$ and $\forall k \in F_2$ and that the family of inverses $L : \mathcal{U} \times F_2 \to F_1$ is a smooth tame map. Then $f$ is locally invertible and each local inverse is a smooth tame map.
What are the Tools from Functional Analysis?

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<td>Nash-Moser</td>
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</table>
What are the Toys we can play with?

Riemannian $\subset$ Symplectic $\subset$ Poisson Geometry

**Riemannian metric** = smoothly varying inner product on a manifold $M$

$$g_x : T_x M \times T_x M \to \mathbb{R}$$

$(U, V) \mapsto g_x(U, V)$

**strong Riemannian metric** = for every $x \in M$, $g_x : T_x M \to (T_x M)^*$

is an isomorphism

**weak Riemannian metric** = for every $x \in M$, $g_x : T_x M \to (T_x M)^*$

is just injective

Levi-Cevita connection may not exist for a weak Riemannian metric.
What are the Toys we can play with?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

**Symplectic form** = smoothly varying skew-symmetric bilinear form

$$\omega_x : T_x M \times T_x M \rightarrow \mathbb{R}$$

$$(U, V) \mapsto \omega_x(U, V)$$

with $d\omega = 0$ and $(T_x M)^\perp_\omega = \{0\}$

**strong symplectic form** = for every $x \in M$, $\omega_x : T_x M \rightarrow (T_x M)^*$

is an isomorphism

**weak symplectic form** = for every $x \in M$, $\omega_x : T_x M \rightarrow (T_x M)^*$

is just injective

Darboux Theorem does not hold for a weak symplectic form
What are the Toys we can play with?

**Riemannian ⊂ Symplectic ⊂ Poisson Geometry**

### Hamiltonian Mechanics

**\((M, g)\) strong Riemannian manifold**

- \(b : T_x M \simeq T^*_x M \quad b^{-1} = \#\)
- \(U \mapsto g_x(U, \cdot)\)

- **Kinetic energy \(=\) Hamiltonian**
  - \(H : T^* M \rightarrow \mathbb{R}\)
  - \(\eta_x \mapsto g_x(\eta^\#, \eta^\#)\)

**\((T^* M, \omega)\) strong symplectic manifold**

- \(\pi : T^* M \rightarrow M\)
- \(\omega = d\theta\)
- \(\theta_{(x, \eta)} : T_{x,\eta} T^* M \rightarrow \mathbb{R}\) \(\eta(\pi_*(X))\) \(Liouville 1\)-form
- geodesic flow \(=\) flow of Hamiltonian vector field \(X_H : dH = \omega(X_H, \cdot)\)
What are the Toys we can play with?

Riemannian $\subseteq$ **Symplectic** $\subseteq$ **Poisson Geometry**

**Poisson bracket** = family of bilinear maps

$\{\cdot, \cdot\}_U : C^\infty(U) \times C^\infty(U) \to C^\infty(U)$, $U$ open in $M$ with

- skew-symmetry $\{f, g\}_U = -\{g, f\}_U$
- Jacobi identity $\{f, \{g, h\}_U\}_U + \{g, \{h, f\}_U\}_U + \{h, \{f, g\}_U\}_U = 0$
- Leibniz rule $\{f, gh\}_U = \{f, g\}_U h + g\{f, h\}_U$

A strong symplectic form defines a Poisson bracket by

$\{f, g\} = \omega(X_f, X_g)$ where $df = \omega(X_f, \cdot)$ and $dg = \omega(X_g, \cdot)$

A Poisson bracket may not be given by a bivector field
What are the Toys we can play with?

Riemannian, Symplectic, Complex \( \subset \ \text{Kähler} \subset \text{hyperkähler Geometry} \)

**Complex structure** = smoothly varying endomorphism \( J \) of the tangent space s.t. \( J^2 = -1 \).

**Integrable complex structure** : s. t. there exists an holomorphic atlas

**Formally integrable complex structure** : with Nijenhuis tensor = 0

**Newlander-Nirenberg Theorem** is not true in general: formal integrability does not imply integrability.
What are the traps of infinite-dimensional geometry?

In infinite-dimensional geometry, the golden rule is: "Never believe anything you have not proved yourself!"

- The distance function associated to a Riemannian metric may be the zero function (Example by Michor and Mumford).
- Levi-Cevita connection may not exist for weak Riemannian metrics.
- Hopf-Rinow Theorem does not hold in general: geodesic completeness $\neq$ metric completeness.
- Darboux Theorem does not apply to weak symplectic forms.
- A formally integrable complex structure does not imply the existence of a holomorphic atlas.
- The tangent space differs from the space of derivations (even on a Hilbert space).
- A Poisson bracket may not be given by a bivector field (even on a Hilbert space).
Abstract. We give a method to construct Poisson brackets \( \{ \cdot, \cdot \} \) on Banach manifolds \( M \), for which the value of \( \{ f, g \} \) at some point \( m \in M \) may depend on higher order derivatives of the smooth functions \( f, g : M \to \mathbb{R} \), and not only on the first-order derivatives, as it is the case on all finite-dimensional manifolds. We discuss specific examples in this connection, as well as the impact on the earlier research on Poisson geometry of Banach manifolds.

1. Introduction

The Poisson brackets in infinite-dimensional setting have played for a long time a significant role in various areas of mathematics including classical mechanics and integrable systems theory (see e.g. [Fad80]). However, the rigorous approach to the notion of Poisson manifold in the context of Banach space is relatively new (see e.g. [AMR02] and [OR03]). It is known that the Poisson brackets on infinite-dimensional manifolds lack some of the properties known from the finite-dimensional case. It was shown for instance in [OR03] that the
\( \mathcal{H} \) separable Hilbert space

**Kinetic tangent vector** \( X \in T_x \mathcal{H} \) equivalence classes of curves \( c(t) \), \( c(0) = x \), where \( c_1 \sim c_2 \) if they have the same derivative at 0 in a chart.

**Operational tangent vector** \( x \in \mathcal{H} \) is a linear map \( D : C^\infty(\mathcal{H}) \to \mathbb{R} \) satisfying Leibniz rule:

\[
D(fg)(x) = Df \ g(x) + f(x) \ Dg
\]
Poisson bracket not given by a Poisson tensor

Ingredients:

- Riesz Theorem
- Hahn-Banach Theorem
- compact operators $\mathcal{K}(\mathcal{H}) \subsetneq \mathcal{B}(\mathcal{H})$ bounded operators

\[ \Rightarrow \exists \ell \in \mathcal{B}(\mathcal{H})^* \text{ such that } \ell(\text{id}) = 1 \text{ and } \ell|_{\mathcal{K}(\mathcal{H})} = 0. \]

Queer tangent vector [Kriegl-Michor]

Define $D_x : C_x^\infty(\mathcal{H}) \to \mathbb{R}$, $D_x(f) = \ell(d^2(f)(x))$, where the bilinear map $d^2(f)(x)$ is identified with an operator $A \in \mathcal{B}(\mathcal{H})$ by Riesz Theorem

\[ d^2(f)(x)(X, Y) = \langle X, AY \rangle \]

Then $D_x$ is an operational tangent vector at $x \in \mathcal{H}$ of order 2.
Poisson bracket not given by a Poisson tensor

Queer tangent vector [Kriegl-Michor]

\[
d(fg)(x)(X) = df(x)(X).g(x) + f(x).dg(x)(X)
\]

\[
d^2(fg)(x)(X, Y) = d^2 f(x)(X, Y).g(x) + df(x)(X)dg(x)(Y) \\
+ df(x)(Y)dg(x)(X) + f(x)d^2 g(x)(X, Y)
\]

\[
d^2(fg)(x) = d^2 f(x).g(x) + \ell(df(x) \otimes dg(x)) \\
+ \ell(dg(x) \otimes df(x)) + f(x)d^2 g(x)(X, Y)
\]

\[
D_x(fg) = \ell(d^2(fg)(x)) \\
= \ell(d^2 f(x)).g(x) + f(x)\ell(d^2 g(x)) \\
+ \ell(df(x) \otimes dg(x)) + \ell(dg(x) \otimes df(x)) \\
= D_x f \ g(x) + f(x) \ D_x g
\]
Theorem

Consider $\mathcal{M} = \mathcal{H} \times \mathbb{R}$. Denote points of $\mathcal{M}$ as $(x, \lambda)$. Then $\{\cdot, \cdot\}$ defined by

$$\{f, g\}(x, \lambda) := \frac{\partial g}{\partial \lambda}(x, \lambda) - \frac{\partial f}{\partial \lambda}(x, \lambda) D_x(g(\cdot, \lambda))$$

a queer Poisson bracket on $\mathcal{H} \times \mathbb{R}$, in particular it can not be represented by a bivector field $\Pi : T^*\mathcal{M} \times T^*\mathcal{M} \to \mathbb{R}$. The Hamiltonian vector field associated to $h(x, \lambda) = -\lambda$ is the queer operational vector field

$$X_h = \{h, \cdot\} = D_x$$

acting on $f \in C_x^\infty(\mathcal{H})$ by $D_x(f) = \ell(d^2(f)(x))$. 
**Pre-shape space** $\mathcal{F} := \{ f \text{ immersion} : \mathbb{S}^1 \to \mathbb{R}^2 \} \subset \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}^2)$

**Shape space** $\mathcal{I} :=$ 1-dimensional immersed submanifolds of $\mathbb{R}^2$
Pre-shape space $\mathcal{F} := \{ f \text{ embedding } : S^2 \to \mathbb{R}^3 \} \subset C^\infty(S^2, \mathbb{R}^3)$

Shape space $\mathcal{I} :=$ 2-dimensional submanifolds of $\mathbb{R}^3$
Shape spaces are non-linear manifolds

Figure: First line: linear interpolation between some parameterized ballerinas, second line: linear interpolation between arc-length parameterized ballerinas.
### Quotient versus Fiber bundle

#### Riemannian metrics

#### Gauge Invariant metrics

<table>
<thead>
<tr>
<th>Group $G$</th>
<th>Some elements of one orbit under the group $G$</th>
<th>a preferred element in the orbit</th>
<th>another choice of preferred element in the orbit</th>
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</thead>
<tbody>
<tr>
<td>$\mathbb{R}^3$ acting by translation</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /> centered curve: $\int_0^1 \left( f_1(s) \right)^2 + \left( f_2(s) \right)^2 ds = \frac{1}{2} \left( 1, 0 \right)$</td>
<td><img src="image3.png" alt="Image" /> curve starting at $\left( 0, 0 \right)$</td>
</tr>
<tr>
<td>$\text{SO}(3)$ acting by rotation</td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /> axes of approximating ellipse aligned</td>
<td><img src="image6.png" alt="Image" /> tangent vector at starting point horizontal</td>
</tr>
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<tr>
<td>$\mathbb{R}^+$ acting by scaling</td>
<td>![Images of scaled shapes]</td>
<td>![Image of length = 1]</td>
<td>![Image of enclosed area = 1]</td>
</tr>
<tr>
<td>Diff$^+([0, 1])$ acting by reparameterization</td>
<td>![Images of reparameterized shapes]</td>
<td>![Image of arc-length parameterization]</td>
<td>![Image of curvature proportional parameterization]</td>
</tr>
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</table>
For $I = [0, 1]$ or $I = \mathbb{Z}/\mathbb{R} \simeq S^1$, the space of smooth immersions

$$C(I) = \bigcap_{k=1}^{\infty} C^k(I) = \{ \gamma \in C^\infty(I, \mathbb{R}^2)/\mathbb{R}^2, \gamma'(s) \neq 0, \forall s \in I \}.$$

is an open set of $C^\infty(I, \mathbb{R}^2)/\mathbb{R}^2$ for the topology induced by the family of norms $\| \cdot \|_{C^k}$, hence a Fréchet manifold.

$$C_1(I) = \{ \gamma \in C(I) : \int_0^1 |\gamma'(s)| \, ds = 1 \}. $$

$$A_1(I) = \{ \gamma \in C(I) : |\gamma'(s)| = 1, \forall s \in I \} \subset C_1(I).$$
Theorem (A.B.T, S.Preston)

The subset $\mathcal{C}_1(I)$ is a tame $C^\infty$-submanifold of $\mathcal{C}(I)$ and $A_1(I)$ is a tame $C^\infty$-submanifold of $\mathcal{C}(I)$, and thus also of $\mathcal{C}_1(I)$. Its tangent space at a curve $\gamma$ is

$$T_{\gamma} A_1 = \{ w \in C^\infty(S^1, \mathbb{R}^2), w'(s) \cdot \gamma'(s) = 0, \quad \forall s \in S^1 \}.$$

Proof: Uses the implicit function theorem of Nash-Moser.

$G(I) = \text{Diff}^+(\mathbb{S}^1)$ is a tame Fréchet Lie group [Hamilton].

Theorem (A.B.T, S.Preston)

The right action $\Gamma: \mathcal{C}(I) \times G(I) \to \mathcal{C}(I)$, $\Gamma(\gamma, \psi) = \gamma \circ \psi$ of the group of reparameterizations $G(I)$ on the tame Fréchet manifold $\mathcal{C}(I)$ is smooth and tame, and preserves $\mathcal{C}_1(I)$. 
Theorem (A.B.T, S.Preston)

Given a curve $\gamma \in C_1(I)$, let $p(\gamma) \in \mathcal{A}_1(I)$ denote its arc-length-reparameterization, so that $p(\gamma) = \gamma \circ \psi$ where

$$
\psi'(s) = \frac{1}{|\gamma'(\psi(s))|}, \quad \psi(0) = 0.
$$

Then $p$ is a smooth retraction of $C_1(I)$ onto $\mathcal{A}_1(I)$.

Theorem (A.B.T, S.Preston)

$\mathcal{A}_1([0, 1])$ is diffeomorphic to the quotient Fréchet manifold $C_1([0, 1])/G([0, 1])$. 

We will consider the 2-parameter family of elastic metrics on $C_1(I)$ introduced by Mio et al.:

$$G^{a,b}(w, w) = \int_0^1 \left( a (D_s w \cdot v)^2 + b (D_s w, n)^2 \right) |\gamma'(t)| \, dt,$$

(2)

where $a$ and $b$ are positive constants, $\gamma$ is any parameterized curve in $C_1(I)$, $w$ is any element of the tangent space $T_\gamma C_1(I)$, with $D_s w = \frac{w'}{|\gamma'|}$ denoting the arc-length derivative of $w$, $v = \gamma'/|\gamma'|$ and $n = v^\perp$.

Since the reparameterization group preserves the elastic metric $G^{a,b}$, it defines a quotient elastic metric on the quotient space $C_1([0, 1])/G([0, 1])$, which we will denote by $\bar{G}^{a,b}$.

$$\bar{G}^{a,b}([w], [w]) = \inf_{u \in T_\gamma \mathcal{O}} G^{a,b}(w + u, w + u)$$
Figure: First line: linear interpolation between some parameterized ballerinas, second line: linear interpolation between arc-length parameterized ballerinas. Geodesic between some parameterized ballerinas with 200 points using Qmap: execution time = 350 s.
Since $\mathcal{A}_1([0, 1])$ is diffeomorphic to the quotient Fréchet manifold $C_1([0, 1])/\mathcal{G}([0, 1])$, we can pull-back the quotient elastic metric $\bar{G}^{a,b}$ to the space of arc-length parameterized curves $\mathcal{A}_1([0, 1])$ and define

$$\tilde{G}^{a,b}(w, w) = G^{a,b}([w], [w]) = \inf_{u \in T_\gamma \mathcal{O}} G^{a,b}(w + u, w + u)$$

where $w$ is tangent to $\mathcal{A}_1([0, 1])$. If $T_\gamma C_1([0, 1])$ decomposes as $T_\gamma C_1([0, 1]) = T_\gamma \mathcal{O} \oplus \text{Hor}_\gamma$, this minimum is achieved by the unique vector $P_h(w) \in [w]$ belonging to the horizontal space $\text{Hor}_\gamma$ at $\gamma$. In this case:

$$\tilde{G}^{a,b}(w, w) = G^{a,b}(P_h(w), P_h(w)),$$  \hspace{1cm} (3)

where $P_h(w) \in T_\gamma C_1([0, 1])$ is the projection of $w$ onto the horizontal space.
**Theorem**

Let $w$ be a tangent vector to the manifold $\mathcal{A}_1([0, 1])$ at $\gamma$ and write $w' = \Phi n$, where $\Phi$ is a real function in $C^\infty([0, 1], \mathbb{R})$. Then the projection $P_h(w)$ of $w \in T_\gamma \mathcal{A}_1([0, 1])$ onto the horizontal space $\text{Hor}_\gamma$ reads $P_h(w) = w - m v$ where $m \in C^\infty([0, 1], \mathbb{R})$ is the unique solution of

$$-rac{a}{b} m'' + \kappa^2 m = \kappa \Phi, \quad m(0) = 0, \quad m(1) = 0$$  \hspace{1cm} (4)

where $\kappa$ is the curvature function of $\gamma$. 
Quotient versus Fiber bundle
Riemannian metrics
Gauge Invariant metrics

**Figure:** Toy example: initial path joining a circle to the same circle via an ellipse. The 5 first shapes at the left correspond to the path at time $t = 0$, $t = 0.25$, $t = 0.5$, $t = 0.75$ and $t = 1$. The right picture shows the entire path, with color varying from red ($t = 0$) to blue ($t = 0.5$) to red again ($t = 1$).

**Figure:** Gradient of the energy functional at the middle of the path depicted in Fig. 3 for $b = 1$ and different values of the parameter $a/b$. 

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Riemannian metrics on Shape spaces of curves and surfaces
**Figure:** Toy example: initial path joining a circle to the same circle via an ellipse. The 5 first shapes at the left correspond to the path at time $t = 0$, $t = 0.25$, $t = 0.5$, $t = 0.75$ and $t = 1$.

**Figure:** Gradient of the energy functional at the middle of the path connecting a circle to the same circle via an ellipse for different values of the eccentricity of the middle ellipse. The first line corresponds to the values of parameters $a = 0.01$ and $b = 1$. The second line corresponds to $a = 100$ and $b = 1$. 
Figure: Gradient of the energy functional along the path depicted in Fig. 3 for $a = 1$ (upper line), $a = 5$ (middle line) and $a = 50$ (lower line) and $b = 1$. 
Figure: 2-parameter family of variations of the middle shape of a path connecting a circle to the same circle
Figure: Energy functional for the 2-parameter family of paths whose middle shape is one of the shapes depicted in Fig. 8. The left upper picture corresponds to $a = 0.01$, $b = 1$ and the right upper picture to $a = 100$, $b = 1$. 
Figure: Energy functional for the 2-parameter family of paths whose middle shape is one of the shapes depicted in Fig. 8.
Genus-0 surfaces of $\mathbb{R}^3$ are Riemann surfaces. Since they are compact and simply connected, the Uniformization Theorem says that they are conformally equivalent to the unit sphere. This means that, given a spherical surface, there exists a homeomorphism, called the uniformization map, which preserves the angles and transforms the unit sphere into the surface. The uniformization maps are parameterized by $PSL(2, \mathbb{C})$. 

Figure: Scalar product on the tangent plan to the tip of the middle finger of a hand.

A.B. Tumpach, *Gauge invariance of degenerate Riemannian metrics*, Notices of AMS.
Figure: Pairs of paths projecting to the same path in Shape space, but with different parametrizations. The energies of these paths, as computed by our program, are respectively (from the upper row to the lower row):

\[ E_\Delta = 225.3565, \quad E_\Delta = 225.3216, \quad E_\Delta = 180.8444, \quad E_\Delta = 176.8673. \]
Figure: Four Paths connecting the same shape but with a parametrization depending smoothly on time. The energy computed by our program is respectively $E_\Delta = 0$ for the path of hands, $E_\Delta = 0.1113$ for the path of horses, $E_\Delta = 0$ for the path of cats, and $E_\Delta = 0.0014$ for the path of Centaurs.


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A.B. Tumpach, Infinite-dimensional hyperkähler manifolds associated with Hermitian-symmetric affine coadjoint orbits, Annales de l’Institut Fourier.

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