Tessellabilities, Reversibilities, and Decomposabilities of Polytopes
— A Survey —

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1. $P_1$-TILES AND $P_2$-TILES
A $P_1$-tile is a polygon which tiles the plane with translations only.
Two families of convex $P_1$-tiles:
(1) parallelograms and
(2) hexagons with three pairs of opposite sides parallel and of the same lengths ($P_1$-hexagons).
A 3-dimensional $P_1$-tile is a polyhedron which tiles the space with translations only.

Five families of **convex** 3-dimensional $P_1$-tiles (Fedorov):

- $F_1$: Parallelepiped (PP)
- $F_2$: Hexagonal Prism (HP)
- $F_3$: Rhombic Dodecahedron (RD)
- $F_4$: Elongated Rhombic Dodecahedron (ERD)
- $F_5$: Truncated Octahedron (TO)
A \( P_2 \)-tile is a polygon which tiles the plane by translations and 180° rotations only.

**Theorem A**
Every convex \( P_2 \)-tile belongs to one of the following four families:

- **\( F_1 \)**: Triangle
- **\( F_2 \)**: Quadrilateral
- **\( F_3 \)**: \( P_2 \)-pentagon (\( BC \parallel ED \))
- **\( F_4 \)**: \( P_2 \)-hexagon (\( QPH \) (\( AB \parallel ED \) and \( |AB|=|ED| \)))
Determine all **convex** 3-dimensional $P_2$-tiles, i.e., convex polyhedra each of which tiles the space in $P_2$-manner. (cf) triangular prism, ...
A net of a convex polyhedron $P$ is defined to be a connected planar object obtained by cutting the surface of $P$.

An ART (almost regular tetrahedron) is a tetrahedron with four congruent faces.
Theorem B (J.A(2007))
Every net (convex or concave) of an ART tiles the plane in $P_2$-manner.
Artworks
Artworks
Artworks
2. REVERSIBILITY
Volvox, a kind of green alga known as one of the most simple colonial (= multicellular) organisms, reproduces itself by reversing its interior offspring and its surface.
Theorem C (J.A. (2007))
If a pair of polygons A and B is reversible, then each of them tiles the plane by translations and 180° rotations only ($P_2$-tiling).

A : red quadrilateral, B: blue triangle
Theorem D (J.A., I. Sato, H. Seong (2013))

For an arbitrary convex $P_2$-tile $P$ and an arbitrary family $F_i$ ($i=1, 2, 3,$ and $4$) of convex $P_2$-tiles, there exists a polygon $Q \in F_i$ such that the pair $P$ and $Q$ is reversible.
A king in a cage
Spider ⇔ Geisha
A 3-dimensional $P_1$-tile is said to be **canonical** if it is **convex** and symmetric with respect to each orthogonal axis.
**Theorem E** (J.A., I. Sato, H. Seong (2011))

For an arbitrary canonical 3-dimensional $P_1$-tile $P$ and an arbitrary family $F_i$ ($i=1, 2, 3, 4, \text{and } 5$) of canonical 3-dimensional $P_1$-tiles, there exists a polyhedron $Q \in F_i$ such that the pair $P$ and $Q$ is reversible.
Cube -> Hexagonal Prism

Hexagonal Prism -> Truncated Octahedron

Rhombic Dodecahedron -> Elongated Rhombic Dodecahedron
3. TILINGS AND ATOMS
Pentadron is a convex pentahedron whose net is as follows:

A symmetric pair of pentadra
Tetrapak is a special kind of ART(tetrahedron with four congruent faces) made by pentadra as follows:
Theorem F (J.A.)
A tetrapak tiles the space and its net tiles the plane.

Problem
Determine all convex polyhedra, each of which tiles the space and one of its nets tiles the plane.
Theorem G (J.A, G.Nakamura, I.Sato (2012))
Every convex 3-dimensional $P_1$-tile (or its affine-stretching transform) can be constructed by copies of a pentadron.
Cube
Hexagonal prism
Truncated octahedron
Rhombic dodecahedron
Elongated rhombic dodecahedron